ELEC 3004/7312: Signals, Systems & Controls
Week 12: Digital Controls Tutorial

This tutorial reviews controls and stability.
Try the sub-problems **first** *without* looking at the Explanation (pp.2-4). There are some tips below.

Problem:

A radar-avoiding cruise missile is designed to fly 10 m above ground-level, at 500 m/s. The missile's ground following radar updates the altimeter every 1/10th of a second, and has a forward obstacle detection range of 1000 m. En route to target, the missile may encounter a 20 m tall obstacle that the missile must ascend to clear. The vertical dynamics of the missile are

$$m\ddot{x} = -c\dot{u} + ku$$

where x is the missile altitude, $m = 1000$ is the vehicle's mass, $c = 1$ is a constant arising from vortex shedding induced by the cruise missile's control surface, $k = 10$ is a constant arising from changing lift of the control surface, and u is the input to the control surface.

a. Derive a continuous transfer function of the plant, from u to x.

b. Digitise the continuous transfer function using modified matched-pole zero method.

c. Plot the transfer function's poles and zeros on the z-plane and draw the system root locus under closed loop feedback.

d. What are the conditions on the pole positions for the missile to clear the obstacle?

e. Sketch the positions of the poles and zeros of a controller that would stabilise altitude under feedback control, and draw the approximate path of the closed loop pole positions with increasing gain.

f. Show that your controller satisfies the conditions needed to clear the obstacle.

g. Why might it be desirable to slow the response of this system?

Tips:

a. What is the Laplace transform of the plant dynamics; Can we solve this for u/x?
b. Remember that for MMPZ we substitute with $(s + a) = (z - e^{-aT})$
c. Consider where the poles and zeros are located.
d. Calculate the practical time available and then consider the settling time.
e. The non-minimum phase zero is key here, in which way will it attract the poles?
f. There are two principal approaches – an analytic one (based on the transfer function) and a graphical approach based on the root locus plots
g. What happens if such a system were to overshoot or undershoot?

Explanation:

a. Use the Laplace transform to convert differential operators to multiples of s:

$$ms^2x = -csu + ku$$

Now, refactor the two sides of the equation to bring u across to the left:

$$\frac{x}{u} = \frac{-cs + k}{ms^2}$$

Expressed with numerical coefficients:

$$\frac{x}{u} = \frac{-s + 10}{1000s^2}$$

b. Digitising with MMPZ method, we make the substitution $(s + a) = (z - e^{-aT})$, where $T = 0.1$, then set the DC gain of the digital system to equal that of the continuous system.
For step one, we will rewrite the system

$$-\frac{1}{1000}\frac{s - 10}{s^2}$$

and substitute for a zero at -10 and poles at the origin:

$$k_{DC}\frac{z - e^1}{(z - e^0)^2}$$

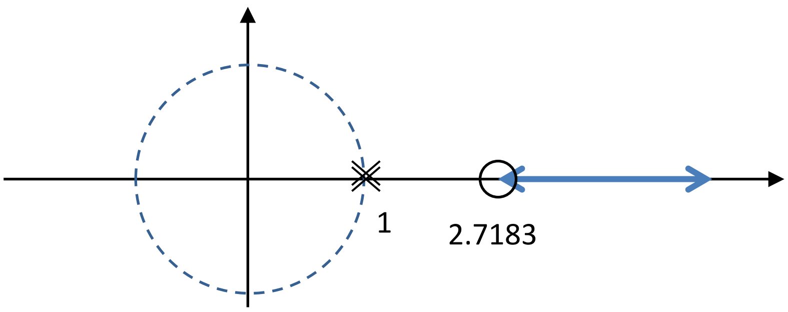
The DC gain k_{DC} can be found by scaling the discrete approximation to match the slope of the continuous 2nd-order slope. Final value theorem says that DC gain of the system will be reached at $z = 1$. We can conceptualise this by multiplying the system by two differentiators, thus ignoring the integrators and considering only the DC gain and zero. It can be seen that the gain will match when:

$$k_{DC}(1 - e^1) = -\frac{0 - 10}{1000}T^2$$

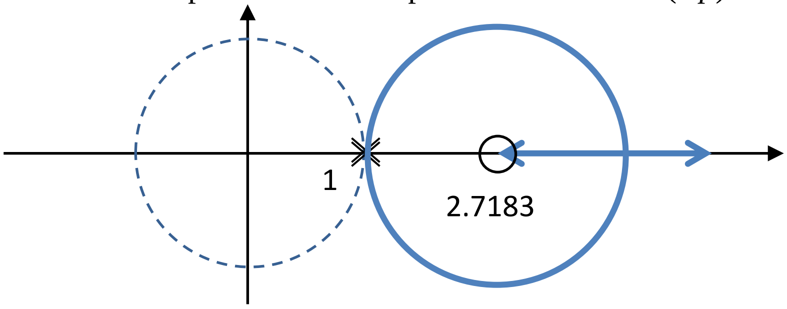
And so $k_{DC} = -5.8198 \times 10^{-5}$
The discrete system is therefore:

$$H(z) = -5.8198 \times 10^{-5}\frac{z - 2.7183}{(z - 1)^2}$$

c. The integrator poles of the system are at z = 1, and the zero is at z = 2.7183. Use pole counting to determine portions of the real axis with root locus. Note the system gain is negative; therefore, starting at the left hand side of the z-plane, the count of poles is zero to the left indicating no locus on the real axis; then crossing the two co-located poles, the count becomes 2, indicating no locus immediately to the right of the poles; crossing the zero the count becomes 1, indicating locus out into the right hand plane.



This second-order system describes a circular locus; the centre of the circle will occur at the zero. Thus, it can be seen that the poles will intercept the real axis at *z*+(*z*-*p*):



d. Travelling at 500 m/s, the missile covers the distance from the first possible detection to the current position in 2 seconds. As the system is sampled at 10 Hz, it is possible for the obstacles to come into view just after a sample is taken, meaning that practical time available is 1.9 seconds. The missile must have reaches peak magnitude and settled within this time.

The settling time is related to the natural frequency and the damping ratio by:

$$\tau_s = \frac{4.6}{\zeta \omega_n}$$

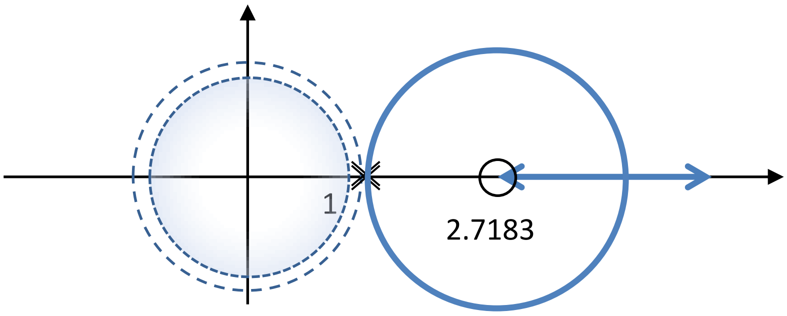
Recall the settling time of a system comes from

$$H(s) = \frac{{\omega_n}^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

which gives time response

$$h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n}(\sin \omega_d\, t)1(t)$$

and $z = e^{sT}$. Substituting for $\zeta\omega_n$, it can be seen that $|z| \leq e^{-T\tau_s}$. a circular bound in the z-plane, within which the system poles must lie:

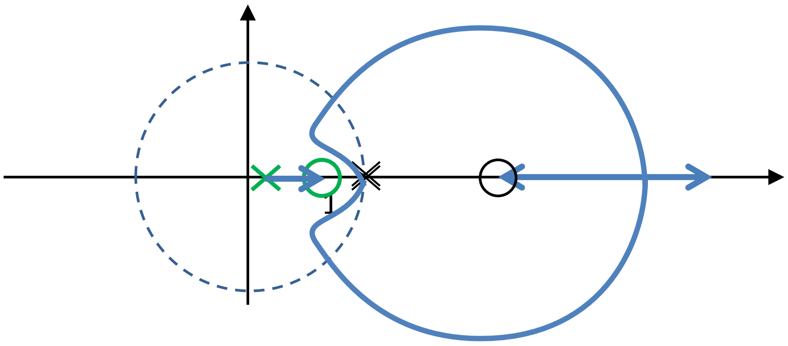


Computing the bound, $|z| \leq 0.82$.

e. The two poles at the origin being tugged to the right by the non-minimum phase zero is the key problem to be solved.

A stable zero is needed to attract the poles to the left. By counting poles it can be seen that adding a zero to the left of the double integrator will put a segment of the root locus on the real axis, while the other pole is absorbed by the unstable zero. In order to keep the controller proper, it should contain a pole; by making this a fast real pole closer to the origin, the centre of the pole-zero system can be moved to the left. This puts locus between the control pole and the control zero, but draw the integrator poles to the left before they go unstable:

Thus, a minimal controller is a lead controller of the form:



Note that a PID controller would put an additional at the origin; makes the control task even more difficult and would not be a good design.

f. There are two approaches to this problem. For the first, assign sane values to the controller: in this case, put the zero at 0.9 and the pole at 0.1:

$$C(z) = k\left(\frac{z-0.9}{z-0.1}\right)$$

The gain is chosen to ensure that all closed loop poles satisfy the performance metric:

$$H_{cl}(z) = \frac{C_n H_n}{C_n H_n + C_d H_d}$$

$$H_{cl}(z) = \frac{-5.8198 \times 10^{-5} k(z-0.9)(z-2.7183)}{-5.8198 \times 10^{-5} k(z-0.9)(z-2.7183) + (z-0.1)(z-1)^2}$$

A little bit of numerical fiddling yields the characteristic polynomial:

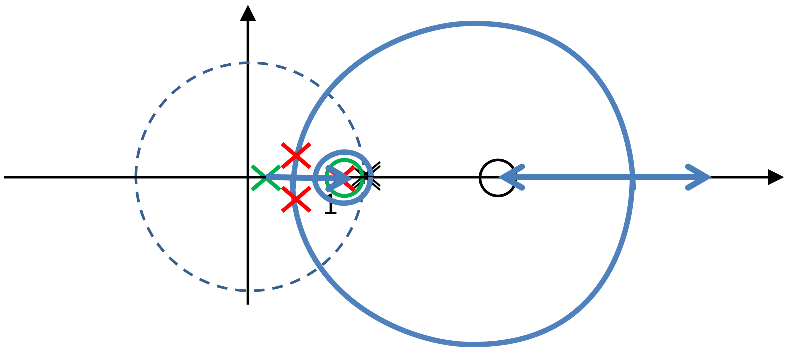
$$z^3 + (-5.8198 \times 10^{-5} k - 2.1)z^2 + (1.2 + 2.1058 \times 10^{-4} k)z - 1.4238 \times 10^{-4} k - 0.1$$

Factorise the polynomial for roots in z, as a function of *k*:

$$\big(z - p_1(k)\big)\big(z - p_2(k)\big)\big(z - p_3(k)\big)$$

Solve $|p_i(k)| < 0.82$ for each root and find the admissible gain magnitude.

The second approach is to recognise that if the zero is placed close to the origin, the poles can be aligned on the real axis, and the zero used to cancel the effective dynamics of the slow pole. As a zero close to the origin induced overshoot, the slow pole-zero cancellation will not cause oscillation that would lead to impacting the obstacle; thus, only the remaining two poles must be carefully placed within the bound; this is easily calculated with the quadratic formula.



g. As the system is non-minimum phase, the system response will include undershoot. By slowing the time response will reduce the amount of undershoot, preventing a possible collision with the ground or objects. By having a near pole-zero cancellation and fast damped poles (around Re(z) = 0.55) , the system can easily clear the obstacle in approximately 1s.