Design of Digital Compensators

or

How I learned to stop worrying and love the z-transform

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Previously on Digital Controls…

- We saw how to approximate continuous dynamic systems with difference equations
- We introduced the z-domain and learned how to convert difference equations to discrete transfer functions
- We saw the relationship between pole positions in the z-plane and characteristic system responses

Feedback control

You guys know how to make a feedback controller, right?

Right…?

... guys?

Anybody?

Two classes of control design

The system…

- Isn't fast enough
- Isn't damped enough
- Overshoots too much
- Requires too much control action

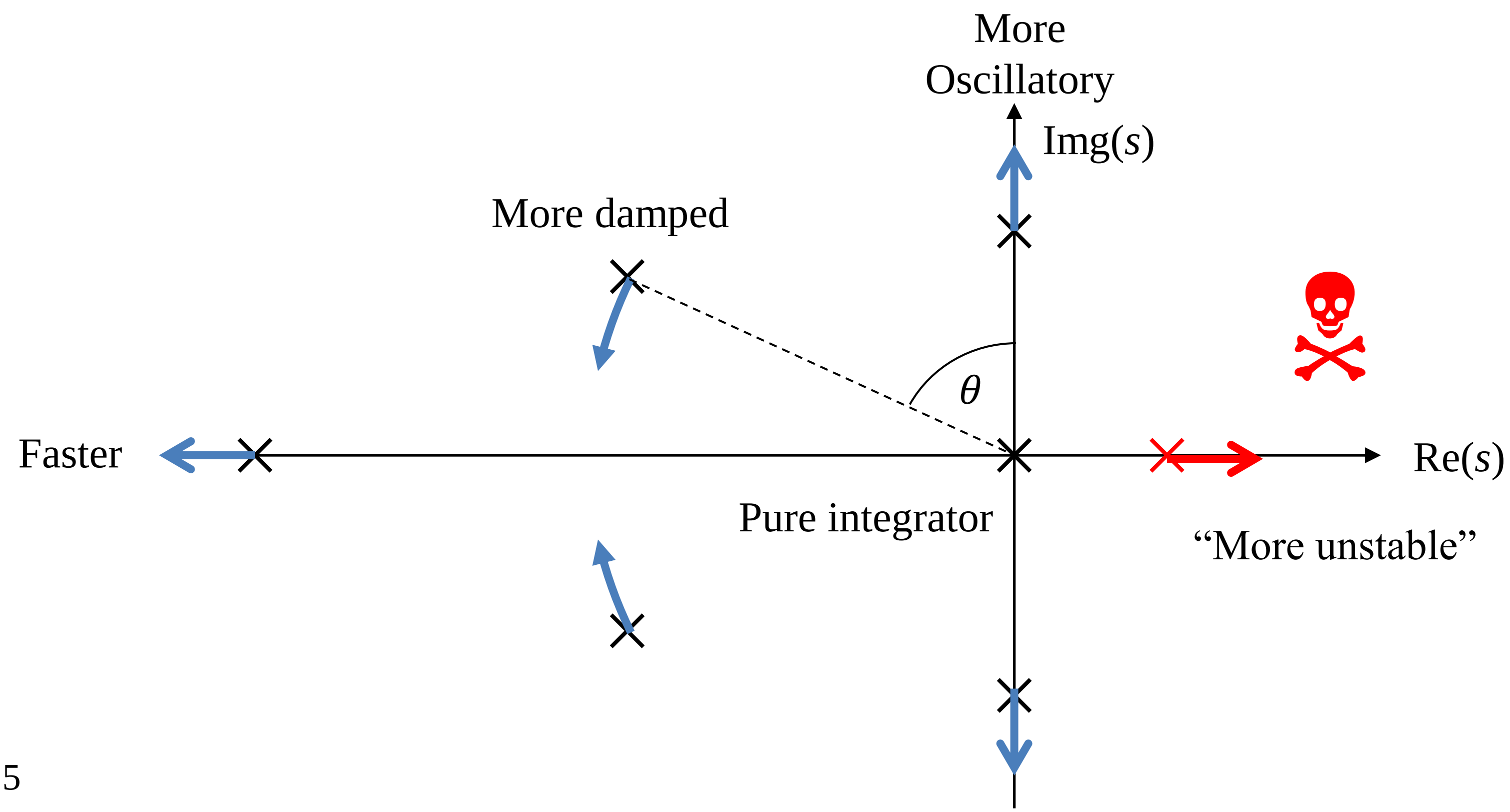
("Performance")

- Attempts to spontaneously disassemble itself

("Stability")

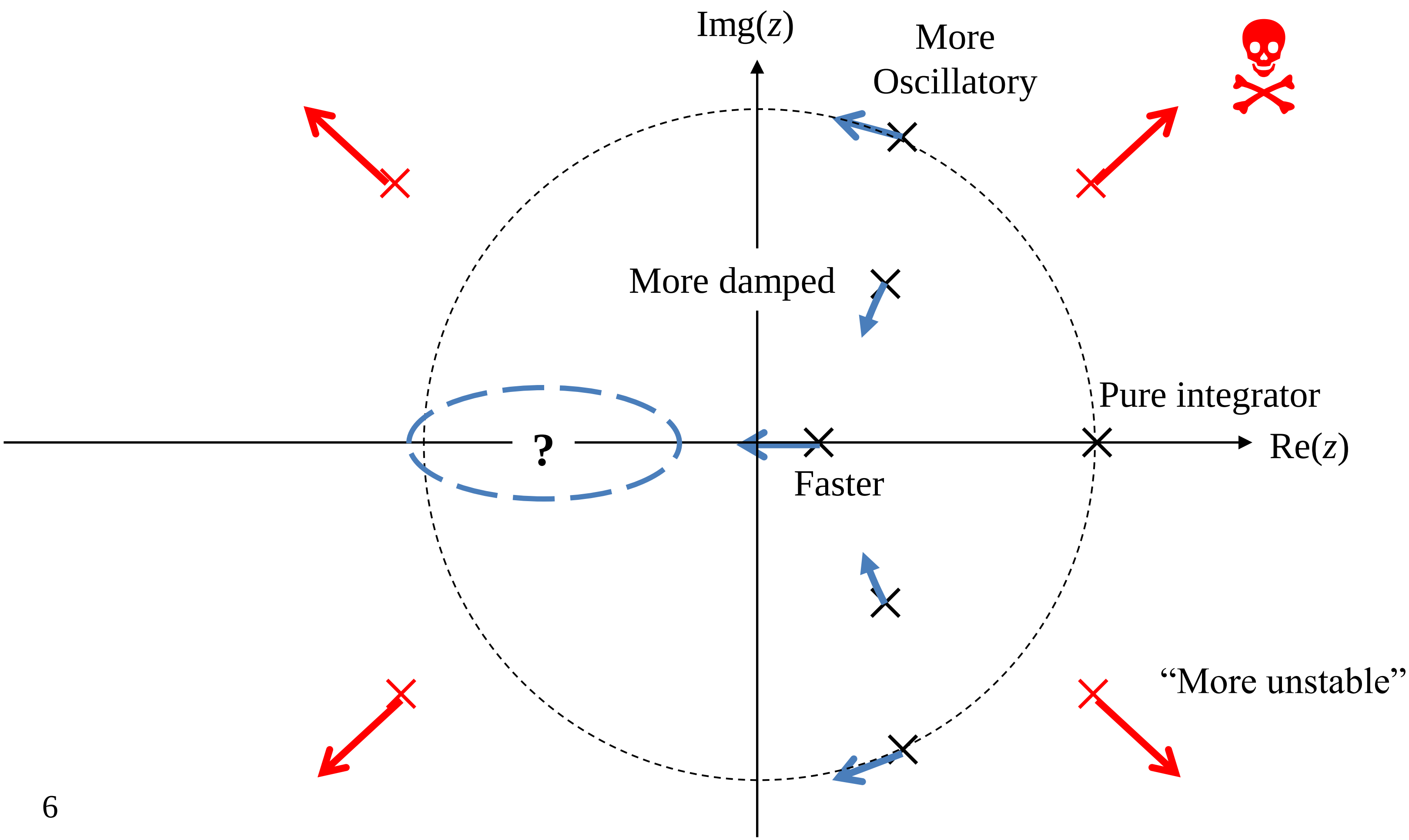
Recall dynamic responses

- Moving pole positions change system response characteristics



Recall dynamic responses

- Ditto the z-plane:



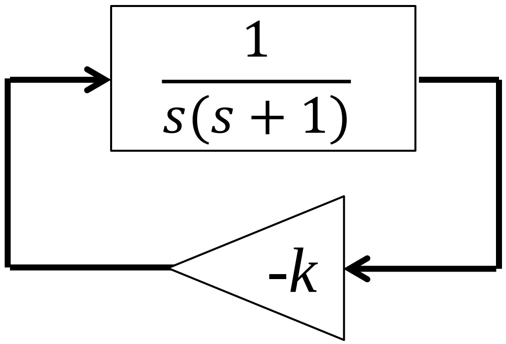
The fundamental control problem

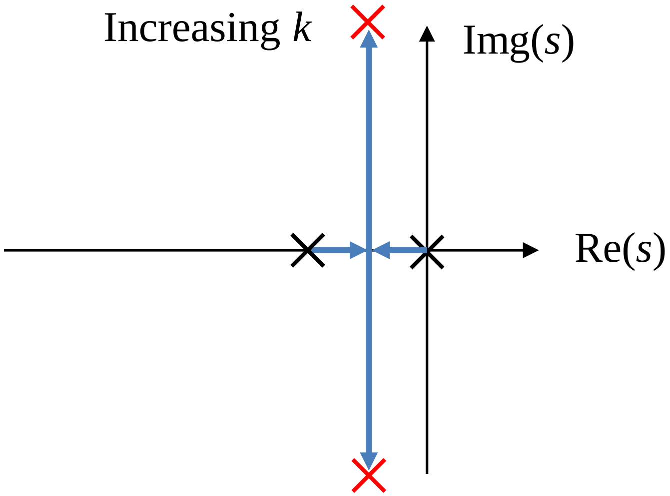
The poles are in the wrong place

How do we get them where we want them to be?

Recall the root locus

- We know that under feedback gain, the poles of the closed-loop system move
  - The root locus tells us where they go!
  - We can solve for this analytically*

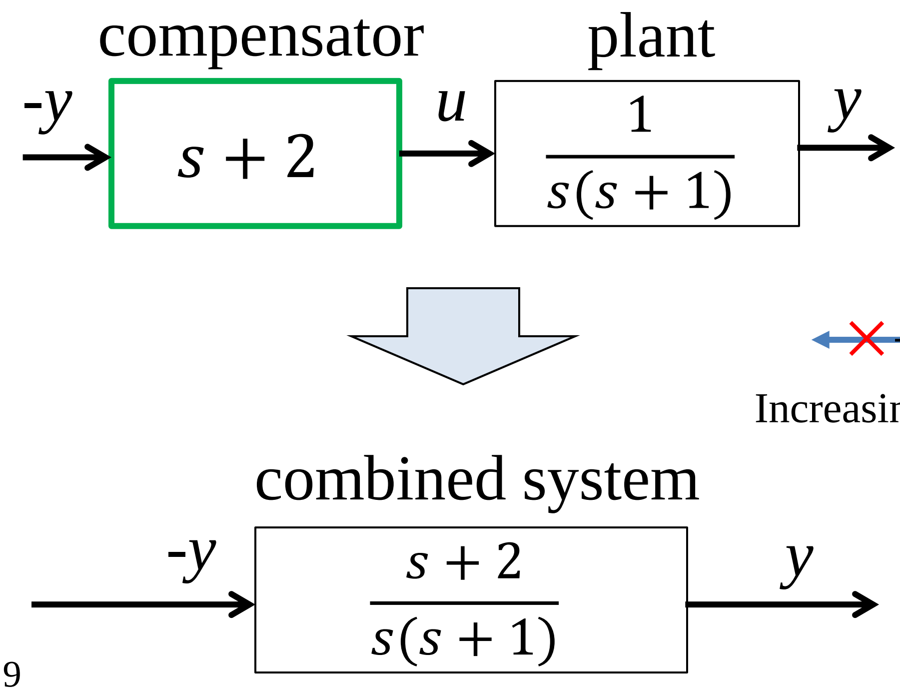


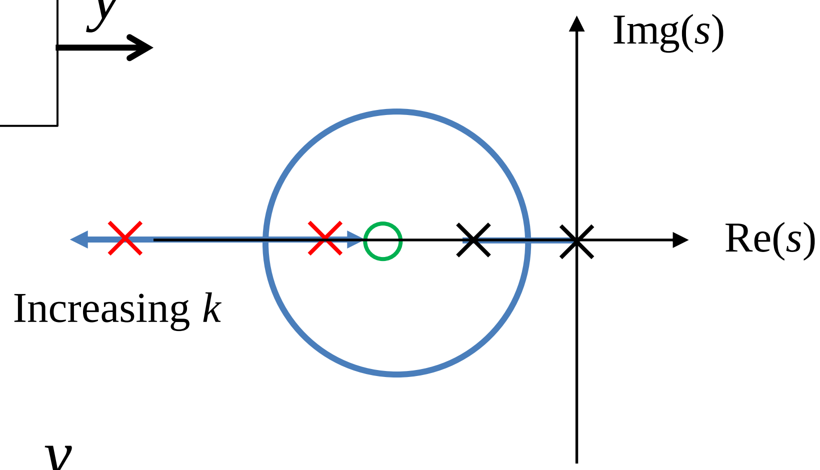


Root loci can be plotted for all sorts of parameters, not just gain!

Dynamic compensation

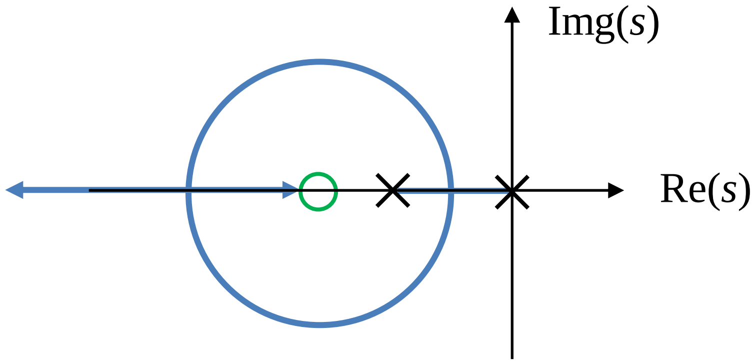
- We can do more than just apply gain!
  - We can add dynamics into the controller that alter the open-loop response





But what dynamics to add?

- Recognise the following:
  - A root locus starts at poles, terminates at zeros
    "Holes eat poles"
  - Closely matched pole and zero dynamics cancel
  - The locus is on the real axis to the left of an odd number of poles (treat zeros as 'negative' poles)



Some standard approaches

- Control engineers have developed time-tested strategies for building compensators
- Three classical control structures:
  - Lead
  - Lag
  - Proportional-Integral-Derivative (PID)

    (and its variations: P, I, PI, PD)

How do they work?

Lead/lag compensation

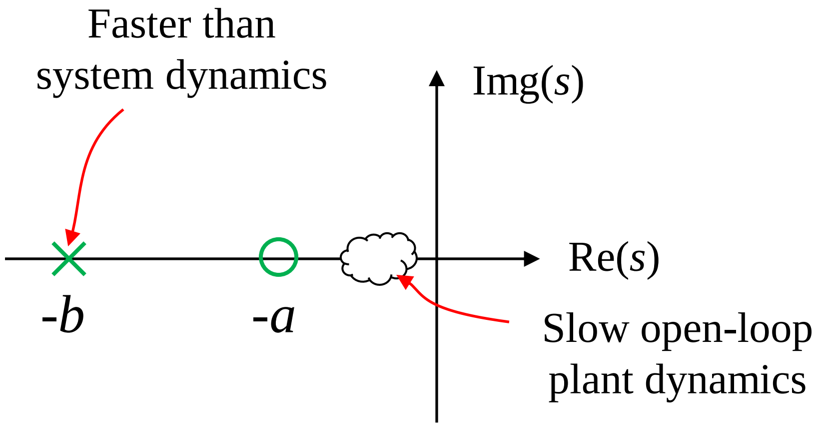
- Serve different purposes, but have a similar dynamic structure:

$$D(s) = \frac{s+a}{s+b}$$

Note:

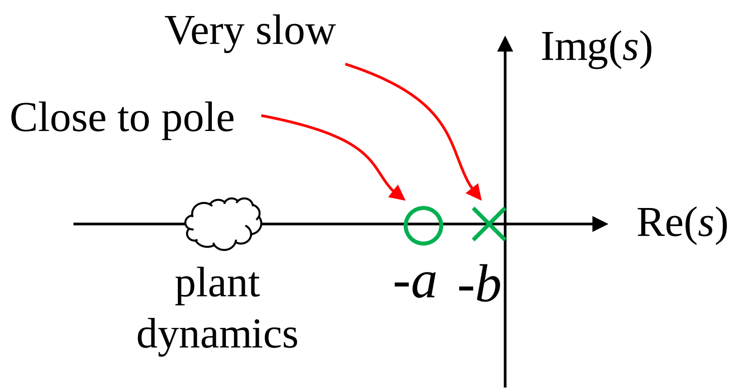
Lead-lag compensators come from the days when control engineers cared about constructing controllers from networks of op amps using frequency-phase methods. These days pretty much everybody uses PID, but you should at least know what the heck they are in case someone asks.

Lead compensation: $a < b$



- Acts to decrease rise-time and overshoot
  - Zero draws poles to the left; adds phase-lead
  - Pole decreases noise
- Set a near desired ω_n; set b at ~3 to 20x a

Lag compensation: $a > b$



- Improves steady-state tracking
  - Near pole-zero cancellation; adds phase-lag
  - Doesn't break dynamic response (too much)
- Set b near origin; set a at ~3 to 10x b

PID – the Good Stuff

- Proportional-Integral-Derivative control is the control engineer's hammer*
  - For P,PI,PD, etc. just remove one or more terms

$$C(s) = k\left(\underbrace{1}_{\text{Proportional}} + \underbrace{\frac{1}{\tau_i s}}_{\text{Integral}} + \underbrace{\tau_d s}_{\text{Derivative}}\right)$$

*Everything is a nail

PID – the Good Stuff

- PID control performance is driven by three parameters:
  - k: system gain
  - τ_i: integral time-constant
  - τ_d: derivative time-constant

You're already familiar with the effect of gain.

What about the other two?

Integral

- Integral applies control action based on accumulated output error
  - Almost always found with P control
- Increase dynamic order of signal tracking
  - Step disturbance steady-state error goes to zero
  - Ramp disturbance steady-state error goes to a constant offset

Let’s try it!

Integral

- Consider a first order system with a constant load disturbance, w; (recall as $t \rightarrow \infty$, $s \rightarrow 0$)

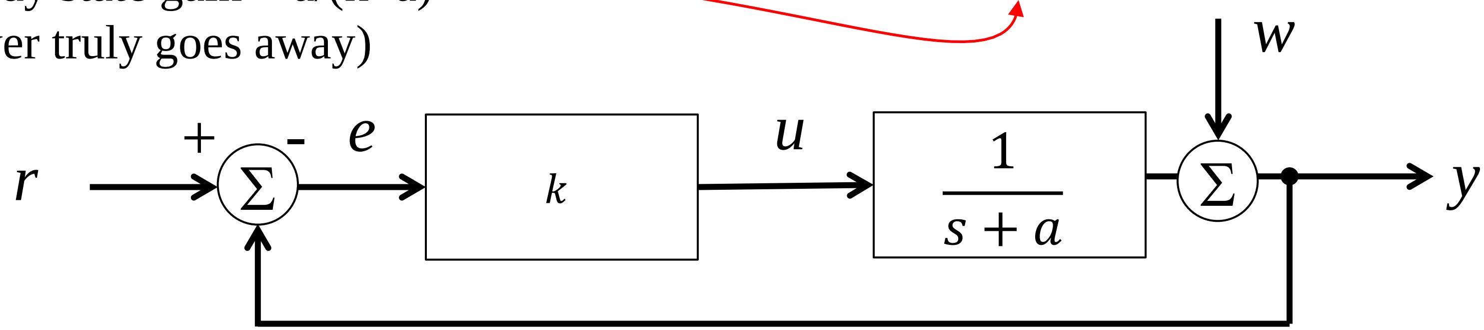
$$y = k\frac{1}{s+a}(r-y)+w$$

$$(s+a)y = k\,(r-y)+(s+a)w$$

$$(s+k+a)y = kr+(s+a)w$$

$$y = \frac{k}{s+k+a}r+\underbrace{\frac{(s+a)}{s+k+a}}w$$

Steady state gain = a/(k+a)
(never truly goes away)



Now with added integral action

$$y = k\left(1 + \frac{1}{\tau_i s}\right)\boxed{\frac{1}{s+a}}(r - y) + w$$

Same dynamics

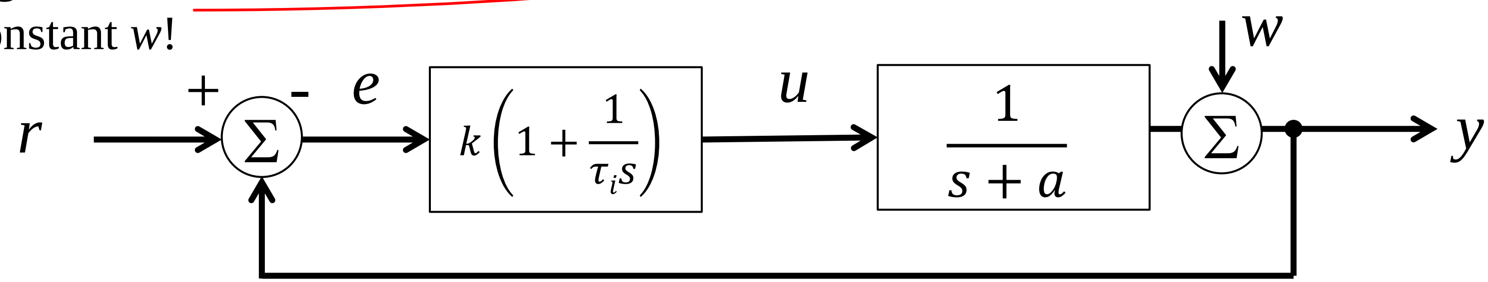
$$y = k\frac{s + \tau_i^{-1}}{s}\frac{1}{s+a}(r - y) + w$$

$$s(s+a)y = k(s + \tau_i^{-1})(r - y) + s(s+a)w$$

$$(s^2 + (k+a)s + \tau_i^{-1})y = k(s + \tau_i^{-1})r + s(s+a)w$$

$$y = \frac{k(s + \tau_i^{-1})}{(s^2 + (k+a)s + \tau_i^{-1})}r + \frac{\boxed{s(s+a)}}{k(s + \tau_i^{-1})}w$$

Must go to zero for constant *w*!



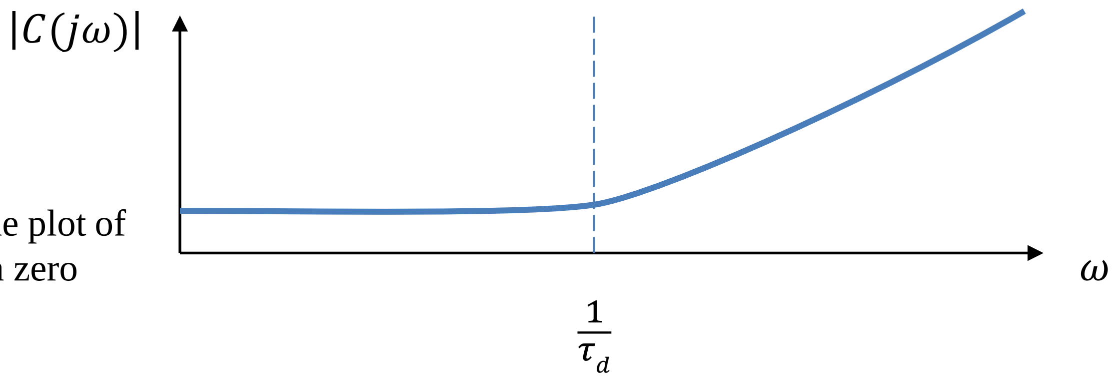
Derivative

- Derivative uses the rate of change of the error signal to anticipate control action
  - Increases system damping (when done right)
  - Can be thought of as 'leading' the output error, applying correction predictively
  - Almost always found with P control*

*What kind of system do you have if you use D, but don't care about position? Is it the same as P control in velocity space?

Derivative

- It is easy to see that PD control simply adds a zero at $s = -\frac{1}{\tau_d}$ with expected results
  - Decreases dynamic order of the system by 1
  - Absorbs a pole as $k \to \infty$
- Not all roses, though: derivative operators are sensitive to high-frequency noise



Bode plot of a zero

PID

- Collectively, PID provides two zeros plus a pole at the origin
  - Zeros provide phase lead
  - Pole provides steady-state tracking
  - Easy to implement in microprocessors
- Many tools exist for optimally tuning PID
  - Zeigler-Nichols
  - Cohen-Coon
  - Automatic software processes

Be alert

- If gains and time-constants are chosen poorly, all of these compensators can induce oscillation or instability.

- However, when used properly, PID can stabilise even very complex unstable third-order systems

Now in discrete

- Naturally, there are discrete analogs for each of these controller types:

Lead/lag: $\dfrac{1-\alpha z^{-1}}{1-\beta z^{-1}}$

PID: $k\left(1+\dfrac{1}{\tau_i(1-z^{-1})}+\tau_d(1-z^{-1})\right)$

But, where do we get the control design parameters from?

The s-domain?

Emulation vs Discrete Design

- Remember: polynomial algebra is the same, whatever symbol you are manipulating:

eg. $s^2 + 2s + 1 = (s + 1)^2$

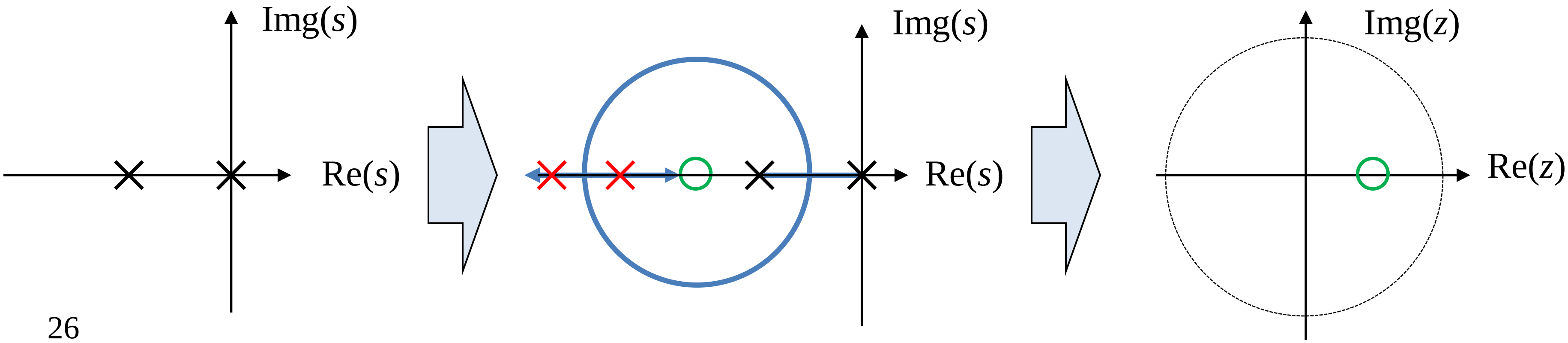
$z^2 + 2z + 1 = (z + 1)^2$

Root loci behave the same on both planes!

- Therefore, we have two choices:
  – Design in the s-domain and digitise (emulation)
  – Design only in the z-domain (discrete design)

Emulation design process

1. Derive the dynamic system model ODE
2. Convert it to a continuous transfer function
3. Design a continuous controller
4. Convert the controller to the z-domain
5. Implement difference equations in software



Emulation design process

- Handy rules of thumb:
  - Use a sampling period of 20 to 30 times faster than the closed-loop system bandwidth
  - Remember that the sampling ZOH induces an effective $T/2$ delay
  - There are several approximation techniques:
    - Euler’s method
    - Tustin’s method
    - Matched pole-zero
    - Modified matched pole-zero

Tustin's method

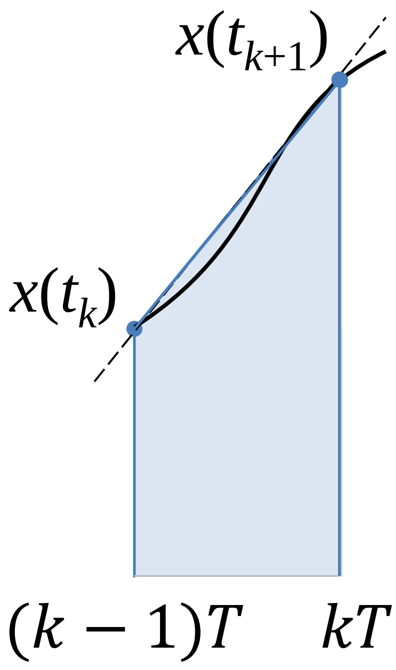
- Tustin uses a trapezoidal integration approximation (compare Euler's rectangles)
- Integral between two samples treated as a straight line:

$$u(kT) = \frac{T}{2}\left[x(k-1) + x(k)\right]$$

Taking the derivative, then z-transform yields:

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

which can be substituted into continuous models



Matched pole-zero

- If $z = e^{sT}$, why can't we just make a direct substitution and go home?

$$\frac{Y(s)}{X(s)} = \frac{s+a}{s+b} \Rightarrow \frac{Y(z)}{X(z)} = \frac{z-e^{-aT}}{z-e^{-bT}}$$

- Kind of!
  - – Still an approximation
  - – Produces quasi-causal system (hard to compute)
  - – Fortunately, also very easy to calculate.

Matched pole-zero

- The process:
  1. Replace continuous poles and zeros with discrete equivalents:

     $$(s + a) \Rightarrow (z - e^{-aT})$$

  2. Scale the discrete system DC gain to match the continuous system DC gain
  3. If the order of the denominator is higher than the enumerator, multiply the numerator by $(1 + z^{-1})$ until they are of equal order*

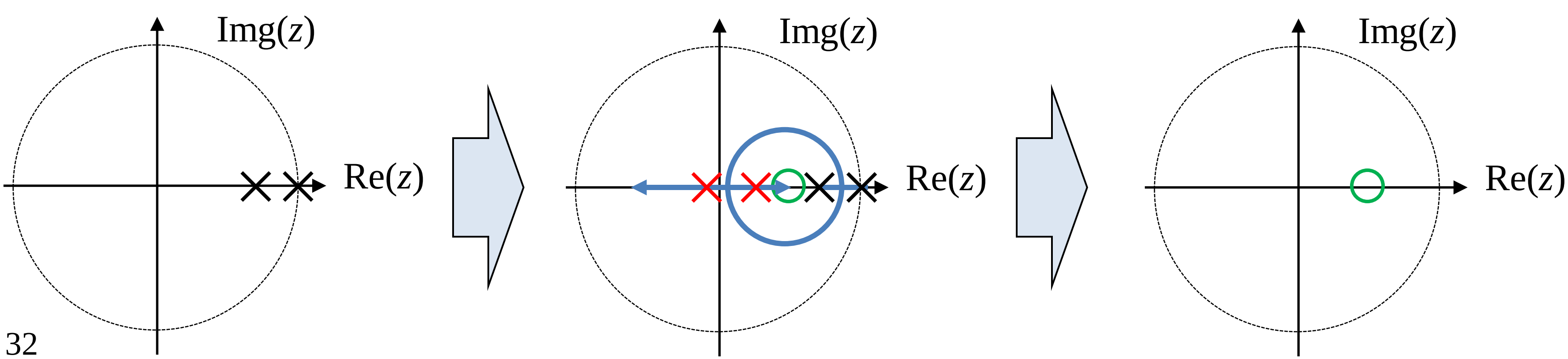
* This introduces an averaging effect like Tustin's method

Modified matched pole-zero

- We're prefer it if we didn't require instant calculations to produce timely outputs
- Modify step 2 to leave the dynamic order of the numerator one less than the denominator
  - Can work with slower sample times, and at higher frequencies

Discrete design process

1. Derive the dynamic system model ODE
2. Convert it to a discrete transfer function
3. Design a digital compensator
4. Implement difference equations in software
5. Pub



Discrete design process

- Handy rules of thumb:
  - Sample rates can be as low as twice the system bandwidth (but 20 to 30 for better performance)
  - A zero at $z = -1$ makes the discrete root locus pole behaviour more closely match the s-plane
  - Beware "dirty derivatives"
    - dy/dt terms derived from sequential digital values are called 'dirty derivatives' – these are especially sensitive to noise!
    - Employ actual velocity measurements when possible

Tune-in next time for…

Digital Design Practice

or

"Why doesn't my compensator work?"

Fun fact: The first UAV was the Hewitt-Sperry Automatic Aeroplane in 1917, 14 years after the Wright bros' first flight.

It could travel 50 km and drop a sandbag within 3.2 km of a target