ELEC 3004/7312: Signals Systems & Controls I
Assignment 1: Solutions

Note: This assignment is worth **10%** of the final course mark. You should spend approximately 3 hours preparing for the tutorial. The tutors will *not* assist you further unless there is real evidence you have attempted questions prior to the tutorial. Beyond the lecture and tutorial sessions, it is estimated that you will need 4 to 5 hours to complete the assignment (**7-8 hours total**).

Total marks: 100

1.**[20]** Even and odd signals refer to their symmetry under time reversal.
An even signal is defined such that it is identical to its time-reverse counterpart.
Consider a sinusoidal source $x(t) = \sin(2\pi ft + \phi)$, configured as a 10Hz band-limited source such that $x(t) = \cos(20\pi t)$. The signal **w[n]** is sampled from **x(t)** at 100 Hz for 10 seconds (for a time vector **t[n]** and a corresponding index vector **n[t]**, written together as **w[n[t]]** or **w[t]**):

a) Is it possible to determine if **w[t]** is even or not? Why?
If so, please describe a simple algorithm for determining if it is even or not
If not, please describe what changes (to constraints, assumptions, or parameters, initial condition, etc.) could be made so as to make this possible.

It is <u>not</u> possible because the initial conditions are not known. Therefore, it cannot be determined if the signal is **<u>symmetric under time reversal</u>** (that is even). What is needed is x[0]. That is, what is missing however **<u>isn't</u>** the series corresponding to –t to 0, just t[0].

Also note that **<u>any signal</u>** can be broken down into a sum of two signals, one of which is even and one of which is odd.

Note that this property is useful in Fourier series and analysis. That time reversal applied to a time signal results in a time reversal of the corresponding sequence of Fourier series coefficients. THUS, if x(t) is odd, then the Fourier Coefficient of the time reversed signal are negative (they are a "odd series").

b) If the frequency, f, of the source signal x(t) was to be increased, what is the maximum bandwidth (frequency) of x(t) for which it is possible to determine if the signal x(t) is even?

Extra Credit: What would it be for w[t]? Consider the aliasing and the phase – what happens if fs/2<f< fs? What happens if fs<f<2fs? (Remember the stroboscopic effect?)

Infinity. x(t) is a **<u>continuous</u>** signal defined on the Real set which spans to +infinity.

Extra Credit (discrete) case
Infinity

For a general source signal (with frequency f0) that is sampled at a sample frequency (fs), we could recall that a uniquely determine a **<u>band-limited signal</u>** is defined on the set that spans (0, fs/2), that is Nyquist. However, this problem is special in that we are not trying to **<u>reconstruct</u>** the signal, just determine a characteristic of its phase. From $f_s/2<f<f_s$ **<u>there is aliasing</u>**, but there system retains phase. From $f_s<f<2f_s$ the phase will be inverted. However, since we know the frequency of the source and the sampling frequency, THEN we know when there is phase reversal and thus can algorithmically compensate for this. Hence Infinity.

2. **[21]** We may describe a system as being:
   - linear[1]
   - time-invariant[2]
   - causal[3]

   Make a 3x7 Matrix in which you determine which of these three properties hold (yes/no) for:

   a) Backward differencer (differentiator): $y[t]=x[t]-x[t-1]$

   b) Forward differencer: $y[t]=x[t+1]-x[t]$

   c) Central differencer: $y(t)=x\left(t+\tfrac{1}{2}\right)-x\left(t-\tfrac{1}{2}\right)$

   d) Multiplier: $y[n]=Gx[n]$

   e) Integrator or Accumulator : $y[n]=\sum_{1}^{n+1}x[k]$

   f) Compressor: $y[n]=x[Mn]$, where M is a positive integer

   g) Square-Law: $y(t)=x^2(t)$

	LTI	**Time Invariant**	**Causal**
A	YES	YES	Yes
B	YES	YES	NO (x[t+1])
C	YES	YES	NO (x[t+1/2])
D	YES	YES	Yes
E	YES	YES	N (Sum to N+1)
F	YES	NO	N (M > 1)
G	NO	YES	Y

3. **[19]** Recall that $Ge^{(a+ib)}=G\left(e^a\right)\left(e^{ib}\right)=G\left(e^a\right)\left(\cos(b)+i\sin(b)\right)$. **Why?**

   Hint: one way, of many, is to consider the Taylor series (about 0) for e^x, sin(x), and cos(x):

   $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots,\ \sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots,\ \cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

   Also, Wikipedia [http://en.wikipedia.org/wiki/Euler%27s_formula] and Wolfram MathWorld [http://mathworld.wolfram.com/EulerFormula.html] have good articles on this.

Please see proof here:
http://mathworld.wolfram.com/EulerFormula.html

[1] Superposition holds, that is for the input/output pairs $\mathbf{x_1[n]} \rightarrow \mathbf{y_1[n]}$ and $\mathbf{x_2[n]} \rightarrow \mathbf{y_2[n]}$, the combination $\mathbf{ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]}$

[2] Characteristics of the system are fixed over time, so for a system with input **x[n]** and output **y[n]**, the input **x[n-m]** will give the output **y[n-m]**

[3] The output only depends on time inputs at the present and past times. The system is nonanticipative.

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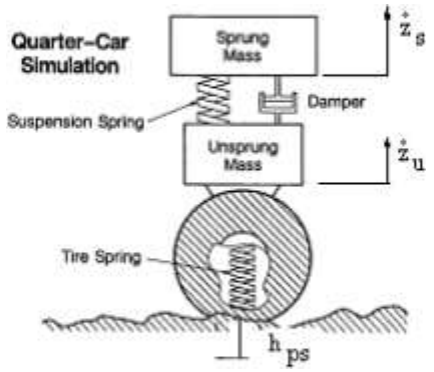
4. [40] Potholes and the Quarter-Car Model

It is desired to measure a pothole from an accelerometer in a car.

Recall that a car's suspension can be viewed as a spring (for the tyre), a mass for the wheel (called the "unsprung mass"), the suspension spring and damper and mass for the car (typically ¼ the vehicle's mass). This is called the quarter-car model. Also recall that an accelerometer is itself a mass connected to a (very stiff) spring that is connected to the outer case (called "ground", but not to be confused with Earth ground or the vehicle chassis).



Quarter-Car Model (from Google Images c/o US DOT http://goo.gl/wOY5m)

a) What is the overall order of the system as seen from the accelerometer (proof mass or case, though please specify) with respect the pothole (assume the ground is infinitely stiff).

4th order … it is two 2nd order systems in parallel.

To prove this, write the ode: (it is not needed to solve the problem)
(for sake of clarity, we define the displacement of the unsprung mass (m_1) as x and displacement of the spring mass (m_2) as y.

$$b\left(\dot{y}-\dot{x}\right)+k_s\left(y-x\right)-k_w\left(x-r\right)=m_1\ddot{x} \tag{1}$$

$$-k\left(y-x\right)-b\left(\dot{y}-\dot{x}\right)=m_2\ddot{y} \tag{2}$$

Some rearranging:

$$\ddot{x}+\frac{b_1}{m_1}\left(\dot{x}-\dot{y}\right)+\frac{k_s}{m_1}\left(x-y\right)+\frac{k_w}{m_1}x=\frac{k_w}{m_1}r \tag{3}$$

$$\ddot{y}+\frac{b_2}{m_2}\left(\dot{y}-\dot{x}\right)+\frac{k_s}{m_2}\left(y-x\right)=0 \tag{4}$$

Take the Laplace (L):

$$s^2X\left(s\right)+s\frac{b}{m_1}\left(X\left(s\right)-Y\left(s\right)\right)+\frac{k_s}{m_1}\left(X\left(s\right)-Y\left(s\right)\right)+\frac{k_w}{m_1}X\left(s\right)=\frac{k_w}{m_1}R\left(s\right) \tag{5}$$

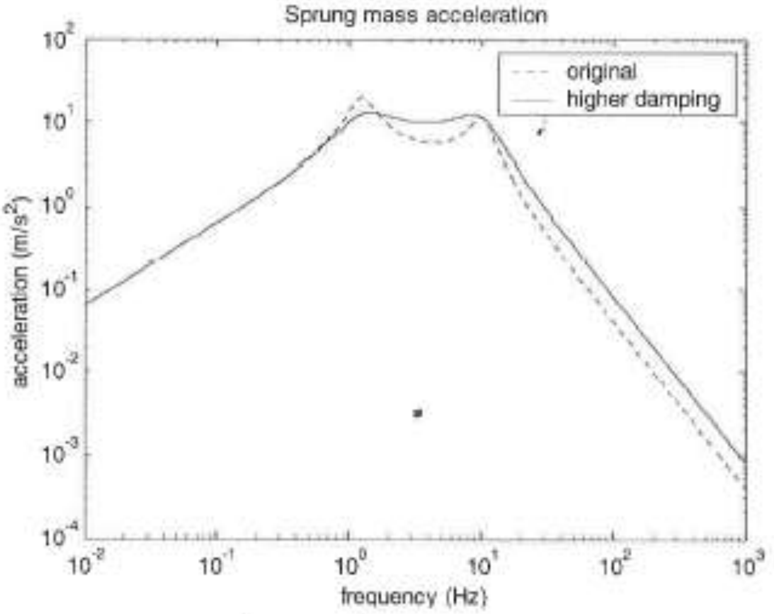
$$s^2Y\left(s\right)+s\frac{b}{m_2}\left(Y\left(s\right)-X\left(s\right)\right)+\frac{k_s}{m_2}\left(Y\left(s\right)-X\left(s\right)\right)=0 \tag{6}$$

More rearranging:

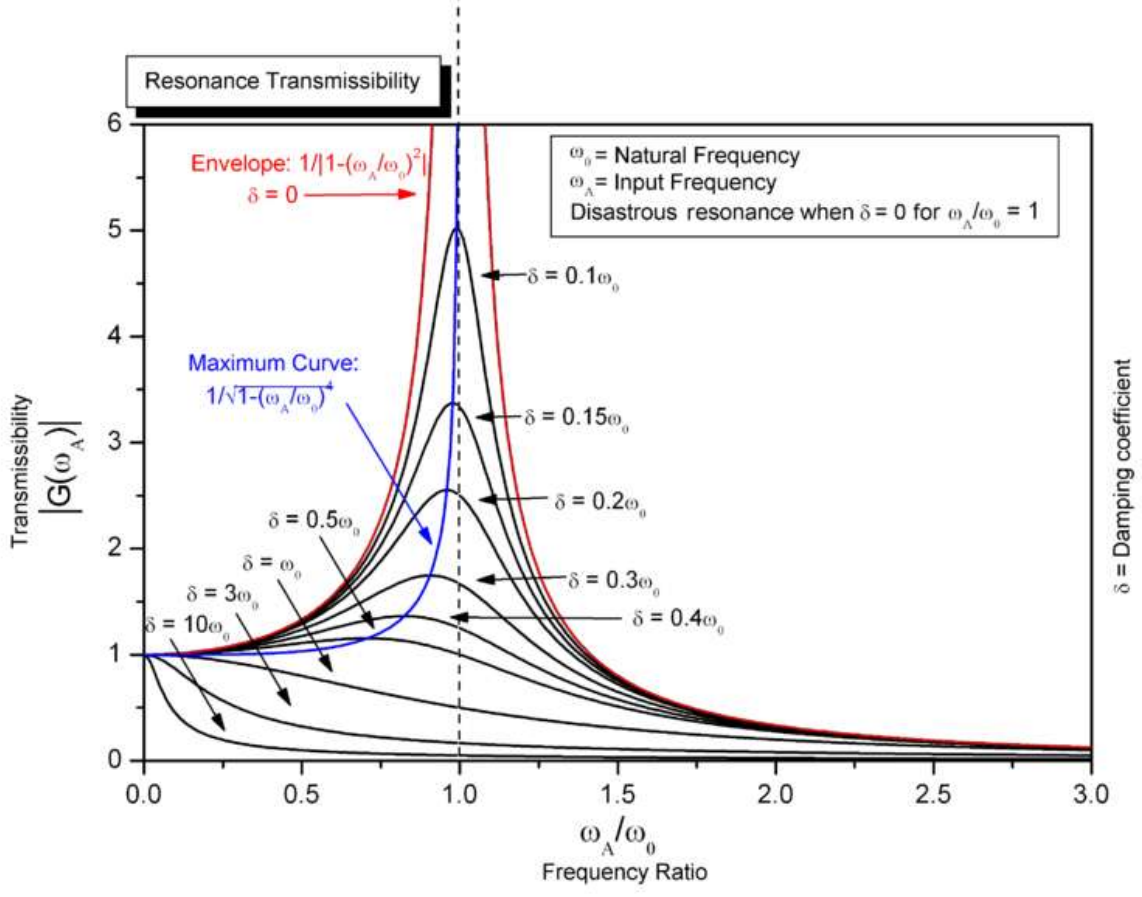
$$\frac{Y(s)}{R(s)}=\frac{\frac{k_wb}{m_1m_2}\left(s+\frac{k_s}{b}\right)}{s^4+\left(\frac{b}{m_1}+\frac{b}{m_2}\right)s^3+\left(\frac{k_s}{m_1}+\frac{k_s}{m_2}+\frac{k_w}{m_1}\right)s^2+\left(\frac{k_wb}{m_1m_2}\right)s+\frac{k_wk_s}{m_1m_2}} \tag{7}$$

b) Sketch (or plot) the expected signal for running over a pothole. Consider the pothole as an impulse, the car's damper as overdamped (shocks are new), and accelerometer as over-damped too (sealed in gas).

It looks something like this (depends on the values one assumes):



Because the system is linear, the output signal is a convolution of the subsystems. Basically two harmonic osillators (or 2nd order systems) together: Recall that this looks like (image form Wikipedia -- http://en.wikipedia.org/wiki/Harmonic_oscillator)



c) Sketch (or plot) the expected signal for dropping an accelerometer on to the floor of the car (presuming, of course, that it is not placed on the floor to begin with and, for example, was placed on top of the dash) with the car not driving over a pothole. Where needed, assume the same as in (b). Is this signal the same as in (b)?

No. It looks something like a second order harmonic oscillator above

d) What sampling rate would you recommend to use (assume an average passenger car, such as a Toyota Corolla)? Consider that excessively high sampling rates use more energy (thus, shorter on-battery operations), give more data to process, and risk more noise. Briefly and succinctly justify your answer based on your research.

20 Hz -- This would capture the dynamic (5-10Hz) of the pothole without too much noise.

Extra Credit: For a system like this (e.g., Street Bump), how might you distinguish the signal from an bump from that of a collision?

There are various ways, including the axis of sensing (vertical instead of horizontal), the frequency content, etc.