2.6 SUMMARY

In this chapter, we have developed important representations for LTI systems, both in discrete time and in continuous time. In discrete time we derived a representation of signals as weighted sums of shifted unit impulses, and we then used this to derive the convolution-sum representation for the response of a discrete-time LTI system. In continuous time we derived an analogous representation of continuous-time signals as weighted integrals of shifted unit impulses, and we used this to derive the convolution integral representation for continuous-time LTI systems. These representations are extremely important, as they allow us to compute the response of an LTI system to an arbitrary input in terms of the system's response to a unit impulse. Moreover, in Section 2.3 the convolution sum and integral provided us with a means of analyzing the properties of LTI systems and, in particular, of relating LTI system properties, including causality and stability, to corresponding properties of the unit impulse response. Also, in Section 2.5 we developed an interpretation of the continuous-time unit impulse and other related singularity functions in terms of their behavior under convolution. This interpretation is particularly useful in the analysis of LTI systems.

An important class of continuous-time systems consists of those described by linear constant-coefficient differential equations. Similarly, in discrete time, linear constant-coefficient difference equations play an equally important role. In Section 2.4, we examined simple examples of differential and difference equations and discussed some of the properties of systems described by these types of equations. In particular, systems described by linear constant-coefficient differential and difference equations together with the condition of initial rest are causal and LTI. In subsequent chapters, we will develop additional tools that greatly facilitate our ability to analyze such systems.

Chapter 2 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

Extension problems introduce applications, concepts, or methods beyond those presented in the text.

BASIC PROBLEMS WITH ANSWERS

2.1. Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \quad \text{and} \quad h[n] = 2\delta[n+1] + 2\delta[n-1].$$

Compute and plot each of the following convolutions:

(a) $y_1[n] = x[n] * h[n]$ **(b)** $y_2[n] = x[n+2] * h[n]$
(c) $y_3[n] = x[n] * h[n+2]$

2.2. Consider the signal

$$h[n] = \left(\frac{1}{2}\right)^{n-1} \{u[n+3] - u[n-10]\}.$$

Express A and B in terms of n so that the following equation holds:

$$h[n-k] = \begin{cases} (\frac{1}{2})^{n-k-1}, & A \leq k \leq B \\ 0, & \text{elsewhere} \end{cases}.$$

2.3. Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2],$$

$$h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.

2.4. Compute and plot $y[n] = x[n] * h[n]$, where

$$x[n] = \begin{cases} 1, & 3 \leq n \leq 8 \\ 0, & \text{otherwise} \end{cases},$$

$$h[n] = \begin{cases} 1, & 4 \leq n \leq 15 \\ 0, & \text{otherwise} \end{cases}.$$

2.5. Let

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 9 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 1, & 0 \leq n \leq N \\ 0, & \text{elsewhere} \end{cases},$$

where $N \leq 9$ is an integer. Determine the value of N, given that $y[n] = x[n] * h[n]$ and

$$y[4] = 5, \quad y[14] = 0.$$

2.6. Compute and plot the convolution $y[n] = x[n] * h[n]$, where

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1] \quad \text{and} \quad h[n] = u[n-1].$$

2.7. A linear system S has the relationship

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-2k]$$

between its input $x[n]$ and its output $y[n]$, where $g[n] = u[n] - u[n-4]$.

(a) Determine $y[n]$ when $x[n] = \delta[n-1]$.
(b) Determine $y[n]$ when $x[n] = \delta[n-2]$.
(c) Is S LTI?
(d) Determine $y[n]$ when $x[n] = u[n]$.

2.8. Determine and sketch the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{elsewhere} \end{cases},$$

$$h(t) = \delta(t+2) + 2\delta(t+1).$$

2.9. Let

$$h(t) = e^{2t}u(-t+4) + e^{-2t}u(t-5).$$

Determine A and B such that

$$h(t-\tau) = \begin{cases} e^{-2(t-\tau)}, & \tau < A \\ 0, & A < \tau < B \\ e^{2(t-\tau)}, & B < \tau \end{cases}.$$

2.10. Suppose that

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

and $h(t) = x(t/\alpha)$, where $0 < \alpha \le 1$.
(a) Determine and sketch $y(t) = x(t) * h(t)$.
(b) If $dy(t)/dt$ contains only three discontinuities, what is the value of α?

2.11. Let

$$x(t) = u(t-3) - u(t-5) \quad \text{and} \quad h(t) = e^{-3t}u(t).$$

(a) Compute $y(t) = x(t) * h(t)$.
(b) Compute $g(t) = (dx(t)/dt) * h(t)$.
(c) How is $g(t)$ related to $y(t)$?

2.12. Let

$$y(t) = e^{-t}u(t) * \sum_{k=-\infty}^{\infty} \delta(t-3k).$$

Show that $y(t) = Ae^{-t}$ for $0 \le t < 3$, and determine the value of A.

2.13. Consider a discrete-time system S_1 with impulse response

$$h[n] = \left(\frac{1}{5}\right)^n u[n].$$

(a) Find the integer A such that $h[n] - Ah[n-1] = \delta[n]$.
(b) Using the result from part (a), determine the impulse response $g[n]$ of an LTI system S_2 which is the inverse system of S_1.

2.14. Which of the following impulse responses correspond(s) to stable LTI systems?
(a) $h_1(t) = e^{-(1-2j)t}u(t)$ **(b)** $h_2(t) = e^{-t}\cos(2t)u(t)$

2.15. Which of the following impulse responses correspond(s) to stable LTI systems?
(a) $h_1[n] = n\cos(\frac{\pi}{4}n)u[n]$ **(b)** $h_2[n] = 3^n u[-n+10]$

2.16. For each of the following statements, determine whether it is true or false:
(a) If $x[n] = 0$ for $n < N_1$ and $h[n] = 0$ for $n < N_2$, then $x[n] * h[n] = 0$ for $n < N_1 + N_2$.
(b) If $y[n] = x[n] * h[n]$, then $y[n-1] = x[n-1] * h[n-1]$.
(c) If $y(t) = x(t) * h(t)$, then $y(-t) = x(-t) * h(-t)$.
(d) If $x(t) = 0$ for $t > T_1$ and $h(t) = 0$ for $t > T_2$, then $x(t) * h(t) = 0$ for $t > T_1 + T_2$.

2.17. Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d}{dt}y(t) + 4y(t) = x(t). \qquad \text{(P2.17–1)}$$

The system also satisfies the condition of initial rest.
(a) If $x(t) = e^{(-1+3j)t}u(t)$, what is $y(t)$?
(b) Note that $\mathcal{R}e\{x(t)\}$ will satisfy eq. (P2.17–1) with $\mathcal{R}e\{y(t)\}$. Determine the output $y(t)$ of the LTI system if

$$x(t) = e^{-t}\cos(3t)u(t).$$

2.18. Consider a causal LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] = \frac{1}{4}y[n-1] + x[n].$$

Determine $y[n]$ if $x[n] = \delta[n-1]$.

2.19. Consider the cascade of the following two systems S_1 and S_2, as depicted in Figure P2.19:

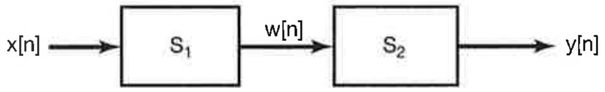

Figure P2.19

S_1 : causal LTI,

$$w[n] = \frac{1}{2}w[n-1] + x[n];$$

S_2 : causal LTI,

$$y[n] = \alpha y[n-1] + \beta w[n].$$

The difference equation relating $x[n]$ and $y[n]$ is:

$$y[n] = -\frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] + x[n].$$

(a) Determine α and β.

(b) Show the impulse response of the cascade connection of S_1 and S_2.

2.20. Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} u_0(t)\cos(t)\,dt$

(b) $\int_{0}^{5} \sin(2\pi t)\,\delta(t+3)\,dt$

(c) $\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)\,d\tau$

BASIC PROBLEMS

2.21. Compute the convolution $y[n] = x[n] * h[n]$ of the following pairs of signals:

(a) $\left.\begin{aligned} x[n] &= \alpha^n u[n], \\ h[n] &= \beta^n u[n], \end{aligned}\right\} \alpha \neq \beta$

(b) $x[n] = h[n] = \alpha^n u[n]$

(c) $x[n] = (-\frac{1}{2})^n u[n-4]$
$h[n] = 4^n u[2-n]$

(d) $x[n]$ and $h[n]$ are as in Figure P2.21.

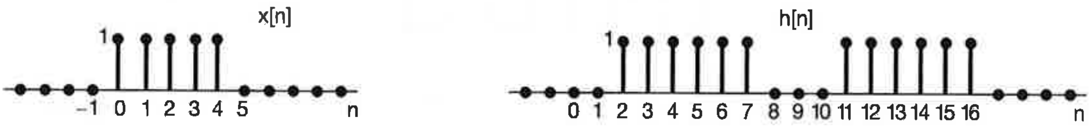

Figure P2.21

2.22. For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

(a) $\left.\begin{aligned} x(t) &= e^{-\alpha t}u(t) \\ h(t) &= e^{-\beta t}u(t) \end{aligned}\right\}$ (Do this both when $\alpha \neq \beta$ and when $\alpha = \beta$.)

Chapter 1 Answers

1.1. $-0.5, -0.5, j, -j, j, 1+j, 1+j, 1-j, 1-j$

1.2. $5e^{j0}, 2e^{j\pi}, 3e^{-j\pi/2}, e^{-j\pi/3}, \sqrt{2}e^{j\pi/4}, 2e^{-j\pi/2}, \sqrt{2}e^{j\pi/4}, e^{j\pi/2}, e^{-j\pi/12}$.

1.3. **(a)** $P_\infty = 0, E_\infty = \frac{1}{4}$ **(b)** $P_\infty = 1, E_\infty = \infty$ **(c)** $P_\infty = \frac{1}{2}, E_\infty = \infty$
(d) $P_\infty = 0, E_\infty = \frac{4}{3}$ **(e)** $P_\infty = 1, E_\infty = \infty$ **(f)** $P_\infty = \frac{1}{2}, E_\infty = \infty$

1.4. **(a)** $n < 1$ and $n > 7$ **(b)** $n < -6$ and $n > 0$ **(c)** $n < -4$ and $n > 2$
(d) $n < -2$ and $n > 4$ **(e)** $n < -6$ and $n > 0$

1.5. **(a)** $t > -2$ **(b)** $t > -1$ **(c)** $t > -2$ **(d)** $t < 1$ **(e)** $t < 9$

1.6. **(a)** No **(b)** No **(c)** Yes

1.7. **(a)** $|n| > 3$ **(b)** all t **(c)** $|n| < 3, |n| \to \infty$ **(d)** $|t| \to \infty$

1.8. **(a)** $A = 2, a = 0, \omega = 0, \phi = \pi$ **(b)** $A = 1, a = 0, \omega = 3, \phi = 0$
(c) $A = 1, a = 1, \omega = 3, \phi = \frac{\pi}{2}$ **(d)** $A = 1, a = 2, \omega = 100, \phi = \frac{\pi}{2}$

1.9. **(a)** $T = \frac{\pi}{5}$ **(b)** Not periodic **(c)** $N = 2$
(d) $N = 10$ **(e)** Not periodic

1.10. π

1.11. 35

1.12. $M = -1, n_0 = -3$

1.13. 4

1.14. $A_1 = 3, t_1 = 0, A_2 = -3, t_2 = 1$

1.15. **(a)** $y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$ **(b)** No

1.16. **(a)** No **(b)** 0 **(c)** No

1.17. **(a)** No; e.g., $y(-\pi) = x(0)$ **(b)** Yes

1.18. **(a)** Yes **(b)** Yes **(c)** $C \le (2n_0 + 1)B$

1.19. **(a)** Linear, not time invariant **(b)** Not linear, time invariant
(c) Linear, time invariant **(d)** Linear, not time invariant

1.20. **(a)** $\cos(3t)$ **(b)** $\cos(3t - 1)$

Chapter 2 Answers

2.1. **(a)** $y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$
(b) $y_2[n] = y_1[n+2]$ **(c)** $y_3[n] = y_2[n]$

2.2. $A = n - 9, B = n + 3$

2.3. $2[1 - \frac{1}{2}^{n+1}]u[n]$

2.4. $$y[n] = \begin{cases} n-6, & 7 \le n \le 11 \\ 6, & 12 \le n \le 18 \\ 24-n, & 19 \le n \le 23 \\ 0, & \text{otherwise} \end{cases}$$

2.5. $N = 4$

2.6. $y[n] = \begin{cases} \frac{3^n}{2}, & n < 0 \\ \frac{1}{2}, & n \geq 0 \end{cases}$

2.7. **(a)** $u[n-2] - u[n-6]$ **(b)** $u[n-4] - u[n-8]$ **(c)** No
(d) $y[n] = 2u[n] - \delta[n] - \delta[n-1]$

2.8. $y(t) = \begin{cases} t+3, & -2 < t \leq -1 \\ t+4, & -1 < t \leq 0 \\ 2-2t, & 0 < t \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

2.9. $A = t - 5, B = t - 4$

2.10. **(a)** $y(t) = \begin{cases} t, & 0 \leq t \leq \alpha \\ \alpha, & \alpha \leq t \leq 1 \\ 1 + \alpha - t, & 1 \leq t \leq 1 + \alpha \\ 0, & \text{otherwise} \end{cases}$ **(b)** $\alpha = 1$

2.11. **(a)** $y(t) = \begin{cases} 0, & -\infty < t \leq 3 \\ \frac{1-e^{-3(t-3)}}{3}, & 3 < t \leq 5 \\ \frac{(1-e^{-6})e^{-3(t-5)}}{3}, & 5 < t \leq \infty \end{cases}$
(b) $g(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)$ **(c)** $g(t) = \frac{dy(t)}{dt}$

2.12. $A = \frac{1}{1-e^{-3}}$

2.13. **(a)** $A = \frac{1}{5}$ **(b)** $g[n] = \delta[n] - \frac{1}{5}\delta[n-1]$

2.14. $h_1(t), h_2(t)$

2.15. $h_2[n]$

2.16. **(a)** True **(b)** False **(c)** True **(d)** True

2.17. **(a)** $y(t) = \frac{1-j}{6}[e^{(-1+3j)t} - e^{-4t}]u(t)$
(b) $y(t) = \frac{1}{6}[e^{-t}(\cos 3t + \sin 3t) - e^{-4t}]u(t)$

2.18. $(1/4)^{n-1}u[n-1]$

2.19. **(a)** $\alpha = \frac{1}{4}, \beta = 1$ **(b)** $[2(\frac{1}{2})^n - (\frac{1}{4})^n]u[n]$

2.20. **(a)** 1 **(b)** 0 **(c)** 0

Chapter 3 Answers

3.1. $x(t) = 4\cos(\frac{\pi}{4}t) - 8\cos(\frac{3\pi}{4}t + \frac{\pi}{2})$

3.2. $x[n] = 1 + 2\sin(\frac{4\pi}{5}n + \frac{3\pi}{4}) + 4\sin(\frac{8\pi}{5}n + \frac{5\pi}{6})$

3.3. $\omega_0 = \frac{\pi}{3}, a_0 = 2, a_2 = a_{-2} = \frac{1}{2}, a_5 = a_{-5}^* = -2j$

3.4. $a_k = \begin{cases} 0, & k = 0 \\ e^{-jk\pi/2}\frac{3\sin(\frac{k\pi}{2})}{k\pi}, & k \neq 0 \end{cases}$