Summary

How the DFT sees its data

How the FT sees its data
How the zero-padded DFT sees its data

But what does the data really do outside the observation interval?
Need for Windows

What if these data look like this?

The DFT sees

Discontinuities

Discontinuities lead to spurious components in DFT
Solution: Data Windowing

Apply a weighting function that forces the data to zero at the end points

Data

* Window

= Windowed Data
Window Functions

- Rectangular (unwindowed)
- Triangular or Bartlett
- Hann (hanning) or raised cosine
- Blackman-Harris

Windows will be discussed when we look at FIR filters.

Vectors as Polynomials

• There is an isomorphism between a vector and the coefficients of a polynomial.
• Thus it is often convenient to replace a vector $v$ with its corresponding polynomial in $z$, say. This is called the Z-transform of $v$, but it is just a clever bookkeeping trick.

$$v = [a_0, a_1, \cdots, a_{n-1}]$$

$$V(z) = a_0 + a_1z^{-1} + \cdots + a_{n-1}z^{n-1}$$
Convolution

Impulse Response

0 1 2
A(0) A(1) A(2) A(3)

Arbitrary Input

b(2) b(1) b(0)

Input applied as it arrives

(2 1 0)

(a(0) (a(1) (a(2) (a(3)

+) +

Output?

Input applied as it arrives

(the magical time reversal)
\[ N^2 \text{ multiplications} \]

<table>
<thead>
<tr>
<th>Conventional integer product</th>
<th>No carry integer product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3</td>
<td>1 2 3</td>
</tr>
<tr>
<td>4 5 6</td>
<td>4 5 6</td>
</tr>
<tr>
<td>7 3 8</td>
<td>6 12 18</td>
</tr>
<tr>
<td>6 1 5</td>
<td>5 10 15</td>
</tr>
<tr>
<td>4 9 2</td>
<td>4 8 12</td>
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<tr>
<td>5 6 0 8 8</td>
<td>4 13 28 27 18</td>
</tr>
<tr>
<td></td>
<td>5 6 0 8 8</td>
</tr>
</tbody>
</table>

Convolution
• We realise that convolution is just integer multiplication without the carry.

Reversed

We can extend this trick and show that polynomial multiplication is equivalent to convolution of the corresponding sequences.
Convolve 1, 2, 3 with 4, 5, 6

1. Convert to polynomials in Z

\[ X(z) = 1 + 2z^{-1} + 3z^{-2} \]
\[ Y(z) = 4 + 5z^{-1} + 6z^{-2} \]

2. Multiply polynomials and gather terms in powers of \( z \)

\[ Z(z) = X(z)Y(z) \]
\[ = (1 + 2z^{-1} + 3z^{-2})(4 + 5z^{-1} + 6z^{-2}) \]
\[ = 4 + 13z^{-1} + 28z^{-2} + 27z^{-3} + 18z^{-4} \]

3. Convert polynomial back to sequence

\[ 4 \ 13 \ 28 \ 27 \ 18 \]
Comments

• Clever bookkeeping trick
  – Recognise the one-to-one and onto correspondence between sequences and polynomials

• Note that multiplication of two polynomials is equivalent to convolution of the corresponding sequences. The power of $z$ in the polynomial corresponds to the output index, and the coefficient corresponds to the output value. The shifting and lining up of powers of $z$ in the product produces the equivalent convolution.
The finite sequence \( h_0, h_1, \cdots, h_{N-1} \) denoted \( \{h_n\} \) has an associated polynomial denoted \( H(z) \) defined by

\[
H(z) = \sum_{n=0}^{N-1} h_n z^{-n}
\]

This polynomial is defined for all values of \( z \). We say that the Region of Convergence (ROC) is the finite plane.
Region of Convergence

• In DSP we generally deal with finite sequences. This means that we do not have to concern ourselves with the region of convergence (ROC) since we can always find the Z transform of a finite sequence and it is well-defined.

• However, sometimes we have to deal with infinite sequences and some infinite sequences do not converge. This is where the ROC is important and the Z transform of such an infinite sequence may only be valid for certain values of $z$. 
The infinite sequence \(h_0, h_1, \ldots, h_N, \ldots\), denoted \(\{h_n\}\), has an associated series denoted \(H(z)\) defined by

\[
H(z) = \sum_{n=0}^{\infty} h_n z^{-n}
\]

This polynomial is not defined for all values of \(z\). We say that the Region of Convergence (ROC) is the annulus containing values of \(z\) for which the above summation is finite.
Example: Infinite Sum

\[ H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \]

\[ = (az^{-1})^0 + (az^{-1})^1 + \cdots + (az^{-1})^{N-1} + \cdots \]

Geometric Series

\[ G = g^0 + g^1 + \cdots \]

\[ gG = g^1 + g^2 + \cdots \]

If \( G \) converges:

\[ G - gG = 1 \]

\[ G(1 - g) = 1 \]

\[ G = \frac{1}{1 - g} \]

Hence

\[ H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \]

\[ \text{Pole at } z=a \]
Region of Convergence

$|z| < 1$
Example: Finite Sum

\[ H(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n \]

\[ = (az^{-1})^0 + (az^{-1})^1 + \cdots + (az^{-1})^{N-1} \]

Geometric Series

\[ S = g^0 + g^1 + \cdots + g^{N-1} \]

\[ gS = g^1 + g^2 + \cdots + g^{N-1} + g^N \]

Hence

\[ S - gS = 1 - g^N \]

\[ S(1 - g) = 1 - g^N \]

\[ S = \frac{1 - g^N}{1 - g} \]

Hence

\[ H(z) = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{z^N - a^N}{z^{N-1}(z - a)} \]

If we let \( N \) go to infinity, we get the infinite series result as long as \( a^N \) goes to zero. That is, \(|a| < 1\).
Direct Convolution

\[ y(n) = \sum_{k=0}^{N-1} x(n-k)h(k) \]

\[ Y(z) = X(z)H(z) = X(z)\sum_{n=0}^{N-1} h(n)z^{-n} \]

Sinusoidal Steady State Response

\[ x(n) = e^{j\phi n} \]

\[ y(n) = \sum_{k=0}^{N-1} e^{j(n-k)\phi} h(k) \]

\[ y(n) = e^{j\phi n}\sum_{k=0}^{N-1} h(k)e^{-jk\phi} \]

For large index \( N \), the summation is the steady state gain \( H(\theta) \)

\[ H(\theta) = \sum_{n=0}^{\infty} h(n)e^{-j\theta n} = H(z) \text{ evaluated at } z = e^{j\theta} \]
Poles and Zeros

- How does all this relate to the poles and zeros in control theory? Are they the same?
- Yes.
- What we are really interested in is representing or approximating systems by polynomial functions.
- The same ideas arise in a number of fields. e.g., mathematics, control, time series forecasting, DSP, electronics. For this reason, the names associated with each technique may vary.
Polynomial Approximations to Functions

• Simple polynomial approximation to a function
  – Taylor series, Power series, Laurent Series, FIR Filter, Moving Average Filter, Non-Recursive Filter, Transversal Filter, Feedforward filter, Linear Phase Filter, All Zero Filter, Tapped Delay Line Filter

• Ratio of two polynomials
  – (Chebyshev-)Pade approximation, IIR Filter, ARMA Filter, Recursive Filter, Feedback Filter, Pole-Zero Filter, General Filter, Rational Transfer Function Model

• Special case where numerator polynomial is a constant
  – AR model, Autoregressive Filter, All Pole, Analog Filter
Different Models

All Zero

\[ X(z) = a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1} \]

Pole Zero

\[ Y(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1}}{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1}} \]

All Pole

\[ Z(z) = \frac{A}{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1}} \]
Difference Equations

\[ y(n) = [b_0 x(n) + b_1 x(n-1) + \cdots + b_M (n-M)] - \]
\[ [a_1 y(n-1) + a_2 y(n-2) + \cdots + a_M y(n-M)] \]

\[ \sum_{k=0}^{M} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \]

\[ Y(z) \sum_{k=0}^{M} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k} \]

\[ T(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{M} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{M-k}}{\sum_{k=0}^{M} a_k z^{M-k}} \]

This step is a bit confusing. It is helpful to remember that this is effectively a system of equations for all \( n \), rather a single equation
Any rational transfer function can be rewritten in the following form. This is called partial fraction expansion.

\[ k_0 \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_M)} \]

\[ = \sum_{k=1}^{M} c_k \frac{z}{z - z_k} \]

The \( c_k \) are called the residues of the system and are in fact the amplitudes of a given exponential response in the total impulse response. The \( z_k \) are the poles of the system.
2 Poles and a Zero

Tent Poles

Thumbtack
The DFT as a Scone Cutter

DFT is the evaluation of the function at equally spaced points around the unit circle.

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Cutting through the foothills of the function.
Summary

- Roots of numerator are called zeros
  - These are the thumbtacks that hold the response down
- Roots of denominator are called poles
  - These are the tent “poles” that push the response up
- Equivalent information
  - Numerator and denominator polynomials
  - Zeros and poles
  - Residues and poles
**FIR Filter Structures**

- If the Z transform has a finite number of weighting terms (numerator polynomial only), they can be stored explicitly within the mechanism which produces the running weighted sum.

Non-recursive

Feedforward

Finite Impulse Response

All Zero
IIR Filter Structures

- If the Z transform has an infinite number of terms (denominator polynomial), we need to have some feedback in the filter.

Infinite impulse response
Recursive
Pole-Zero