LTI & Laplace Transforms

ELEC 3004: Signals, Systems & Control

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Schedule of Events

Week	Date	Lecture Title
1	1-Mar	Overview
2	5-Mar	Signals & Systems
	8-Mar	Sampling
3	12-Mar	LTI & Laplace Transforms
	15-Mar	Convolution
4	19-Mar	Discrete Fourier Series
	22-Mar	Fourier Transform
5	26-Mar	Fourier Transform Operations
	29-Mar	Applications: DFFT and DCT
6	2-Apr	Exam 1 (10%)
	5-Apr	(Guest Lecture from Industry)
7	16-Apr	Data Acquisition & Interpolation
	19-Apr	Noise
8	23-Apr	Filters & IIR Filters
	26-Apr	FIR Filters
9	30-Apr	Multirate Filters
	3-May	Filter Selection
10	7-May	Holiday
	10-May	Quiz (10%)
11	14-May	z-Transform
	17-May	Introduction to Digital Control
12	21-May	Stability of Digital Systems
	24-May	Estimation
13	28-May	Kalman Filters & GPS
	31-May	Applications in Industry

Overview

- Laplace transform
  - Finite power signals
  1. Unilateral Laplace transform
  2. Bilateral Laplace transform
- Transform Analysis of Linear systems
  - Circuit Analysis
  - Transfer functions

Laplace Transform

- Problem: FT of a signal may not always exist!
  - finite power (and not periodic),
  - e.g., $x(t) = u(t)\exp(-at)$ with $a < 0$
  - Or $x(t) = u(t)\cos(5t)$!
- Solution: Force signal to have finite energy
  - Multiply by convergence factor ($\exp(-\sigma t)$)
  - i.e., new signal $x_\sigma(t) = \exp(-\sigma t)x(t)$
  - Therefore, FT of $x_\sigma(t)$ exists

$$X_\sigma(w) = \int_{-\infty}^{\infty} x_\sigma(t)\exp(-jwt)dt$$

Fourier

⇓

Rearranging...

$$X_\sigma(w) = \int_{-\infty}^{\infty} x(t)\exp(-(\sigma + jw)t)dt$$

Laplace

Bilateral Laplace Transform

- For compactness we write
  - $s = \sigma + jw$

Note:

$$X(s) = \int_{-\infty}^{\infty} x(t)\exp(-st)dt$$

Laplace transform
$X(s) = L\{x(t)\}$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\exp(st)ds$$

Inverse Laplace transform
$x(t) = L^{-1}\{X(s)\}$

$$x(t) = \exp(\sigma t)\, x_\sigma(t)$$

i.e., × by inverse of converge factor

$$= \exp(\sigma t)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\sigma(\omega)\exp(j\omega t)\, d\omega$$

i.e., $F^{-1}\{X_\sigma(\omega)\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\sigma(\omega)\exp((\sigma + j\omega)t)\, d\omega$$

Finally re-arrange

Unilateral Laplace Transform

- Problem with *two-sided* (bilateral) transform
  - choice of σ can cause ambiguities in $L^{-1}\{X(s)\}$
    - i.e., as different $x(t)$'s have same $X(s)$!
    - *e.g.*, $L\{\exp(at)u(t)\} = L\{-\exp(at)u(-t)\}$
- Solution: assume $x(t)$ to be causal (*one-sided*)
  - i.e., $x(t) = 0$ when $t < 0$
  - This is termed the *unilateral* Laplace Transform
  - Integration is now from $0 \leq t < \infty$
- This solution works for *most* practical signals
  - Bilateral required for (non-deterministic) random signals
    - (see later)

Unilateral Laplace Transform

- One-sided Laplace transform

$$X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)dt$$

0⁻ indicates origin is included in integration $0 \le t < \infty$

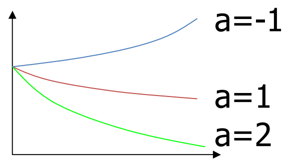
- Laplace transform
  - $X(s) = L\{x(t)\}$
- Inverse Laplace transform
  - $x(t) = L^{-1}\{X(s)\}$

- Consider signal

Convergence of Laplace Transform

$$x(t) = \exp(-at)u(t)$$

$$\begin{aligned} X(s) &= \int_0^{\infty} \exp(-(s+a)t)dt \\ &= \int_0^{\infty} \exp(-(\sigma+a)t)\exp(-jwt)dt \\ &= \frac{-1}{s+a}\left[\exp(-(s+a)t)\right]_0^{\infty} \\ &= \frac{1}{(\sigma+a)+jw}, \quad \sigma+a>0 \\ X(s) &= \frac{1}{s+a}, \quad \Re\{s\} > -a \end{aligned}$$



Effectively same as
← Fourier Transform

- Convergence dependent on both σ and *a*
  - Note: $\Re\{s\} = \sigma$
- Region of Convergence (ROC)
  - Finite integral (energy)

Laplace Examples

Unit step function:

$$L\{u(t)\} = \int_0^\infty u(t)\exp(-st)dt$$
$$= \int_0^\infty \exp(-st)dt$$
$$= \left[-\frac{\exp(-st)}{s}\right]_0^\infty = \frac{1}{s}, \quad \sigma > 0$$

Impulse function:

$$L\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\exp(-st)dt$$
$$= \int_{0^-}^{0^+} \delta(t)\exp(-st)dt$$

Remember:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \int_{0^+}^{0^+} f(t)\delta(t)dt = f(0)\int_{0^+}^{0^+}\delta(t)dt = f(0) \quad = \quad \exp(-s0)\int_{0^-}^{0^+}\delta(t)dt = 1$$

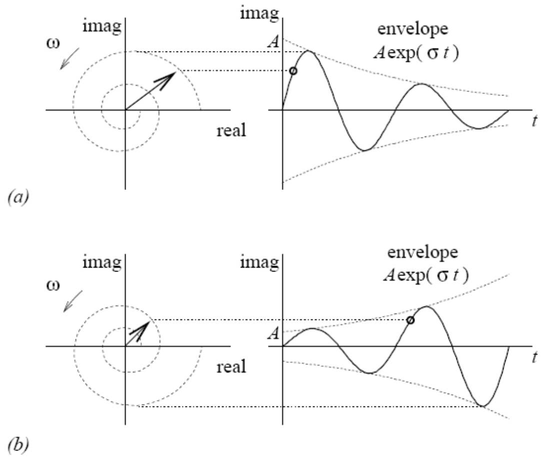
Interpretation of Laplace Transform

- Represents signals, $x(t)$, as sum of
  - growing/decaying cosine waves
- at all frequencies (continuous), $X(s)$
  - $\exp(\sigma t)|X(s)|dw/2\pi$ is amplitude of growing/decaying cosine wave
    - In frequency band w to $w + dw$
  - $\angle X(s)$ is phase shift of cosine wave
- parameter σ ($\Re\{s\}$) determines rate of growth or decay
  - Note: $\sigma = 0$ is the Fourier Transform ☺
    - Constant amplitude cosine waves

Complex Phasors

constant magnitude σ = 0
As per Fourier Transform

$$A \exp(st) = A \exp(\sigma t) \cos(\omega t) + j\; A \exp(\sigma t) \sin(\omega t)$$



Decaying σ < 0

Growing σ > 0

Linear Transforms

- So far, we have looked at
  - Fourier series
    - Trigonometrical & Complex
  - Fourier transform
  - Laplace transform
- All represent signals as a
  - Weighted sum (or integration) of
  - Complex exponentials (that are orthogonal
  - e.g., complex FS, $x(t) = \Sigma\, X_n \exp(jnw_0t)$
- This relates directly to linear systems

Complex Fourier series:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(\, jn\omega_0 t\,)$$

Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(\, j\omega t\,)\, d\omega$$

Laplace transform:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \exp(st)\, ds$$

Linear Transforms & Linear Systems

Useful, due to two properties of linear systems

1. Superposition principle
   - Inputs are complex exponentials (sinusoids)
   - Output is same exponentials, but with different weights, i.e., delay and amplify/attenuate input

$$H\{ax_1(t)+bx_2(t)\} = aH\{x_1(t)\}+bH\{x_2(t)\}$$

In other words, each phasor can be considered individually and output calculated by summation of the individual phasors

Linear Transforms & Linear Systems

2. Ordinary differential equations (ODE)
   - Output *y*(*t*) related to input *x*(*t*) by ODE

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

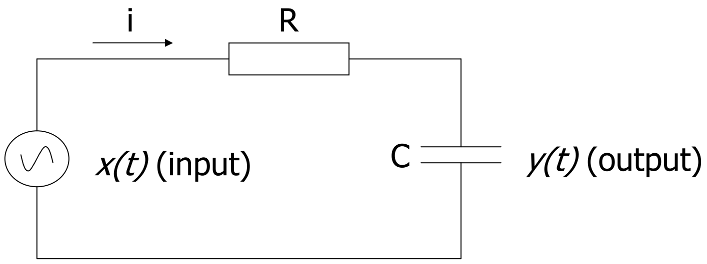
$$\frac{d}{dt}(\exp(jn\omega_0 t)) = jn\omega_0 \exp(jn\omega_0 t)$$

$$\frac{d^2}{dt^2}(\exp(jn\omega_0 t)) = (jn\omega_0)^2 \exp(jn\omega_0 t)$$

Differentiation is simple:
- Frequency unchanged
- Magnitude changes (complex)

Note: initially consider harmonically related sinusoids $n\omega_0$ as per Fourier Series

Example RC Circuit



Using Kirchhoff's laws, system described (modelled) by ODE

(1 energy store ∴ 1st order)

$$a_0 y(t) + a_1 \frac{dy}{dt} = b_0 \exp(jn\omega_0 t)$$

x(t) sinusoidal input

$$y(t) + RC\frac{dy}{dt} = \exp(jn\omega_0 t)$$

Specifically:
$a_0 = 1$
$a_1 = RC$
$b_0 = 1$
derivation shortly...

Example RC Circuit

Note: $\frac{dy_n(t)}{dx} = (jn\omega_0) y_n(t)$

- If input single sinusoid $x_n(t) = \exp(jnw_0t)$
  - and system linear, (steady-state) output has form:

$$y_n(t) = K \exp(jn\omega_0 t)$$

where *K* is <u>complex</u> constant (weight)

Same frequency
Different amp.
&/or phase

- Substituting (assumed) solution into ODE

$$a_0 y_n(t) + a_1 (jn\omega_0) y_n(t) = b_0 \exp(jn\omega_0 t)$$

$$y_n(t) = \frac{b_0}{a_0 + (jn\omega_0)a_1} \exp(jn\omega_0 t)$$

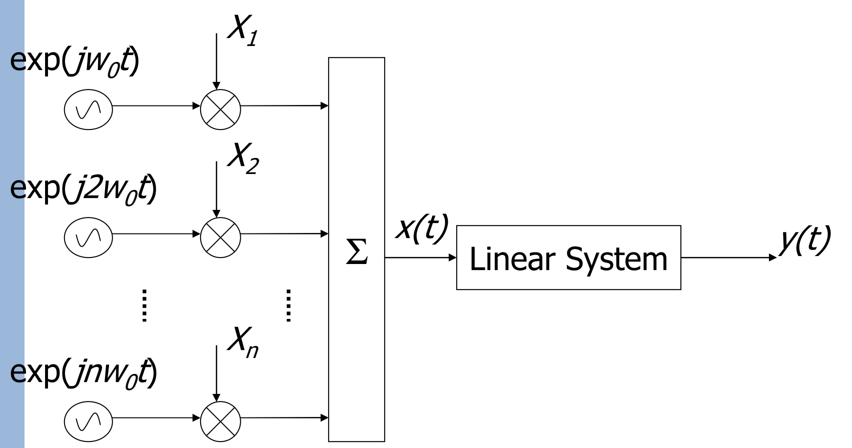
$$y_n(t) = H_n \exp(jn\omega_0 t)$$

$$H_n = \frac{b_0}{a_0 + (jn\omega_0)a_1} = \frac{1}{1 + (jn\omega_0)RC}$$

Where H_n is the system **transfer function** to this single phasor (K)

Response of Linear System

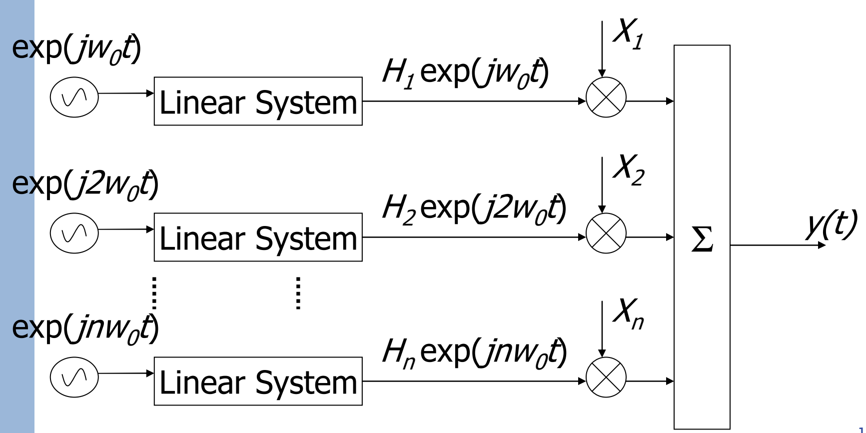
- Input represented as Fourier series



Response of Linear System

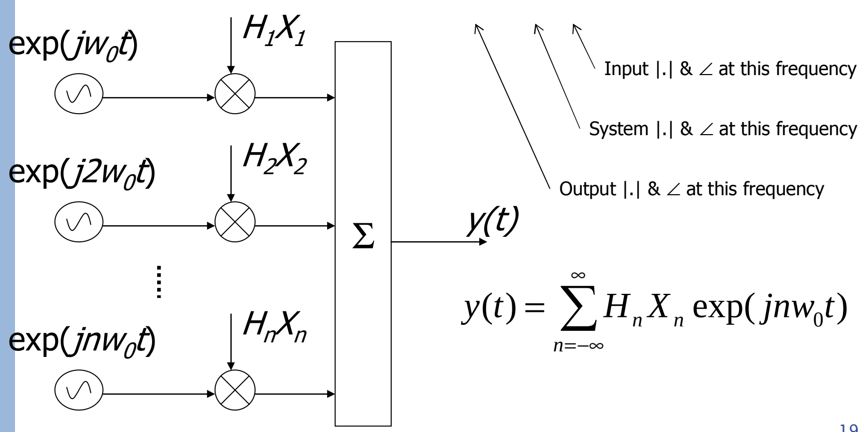
Each phasor is orthogonal and ∴ can be evaluated separately

- Applying superposition principle
  - Applying individual phasors & shifting X_n terms



Response of Linear System

- Output represented as Fourier series
  - output FS coefficients $Y_n = H_n X_n$



$$y(t) = \sum_{n=-\infty}^{\infty} H_n X_n \exp(jnw_0 t)$$

General Approach

- Represent input as weighted sum of
  - Complex exponential **basis functions**,
  - e.g., FS: exp($jn\omega_0 t$): sinusoids at harmonic frequencies
- Basis functions are orthogonal
  - Amplitude (& phase) at each freq. evaluated separately
- System described by ODE
  - Response to exponentials (differentials) easy to calculate
    - Frequency remains constant, Amp. & Phase change
  - e.g., if response 2nd harmonic is $2\omega_0 \exp(2j\omega_0)$
- System is linear
  - Output is sum of responses of individual exponentials
    - i.e., sum response at each frequency

Laplace Transforms

- General approach was illustrated with
  - Periodic input (i.e., we used Fourier series)
  - Steady state response
- But, in general interested in
  - non-periodic input and
  - Both transient & steady state response
- So, we use the Laplace transform
  - Basis functions exp(st), where $s = \sigma + jw$
  - output response is H(s) exp(st)
  - where $H(s)$ is system **transfer function**

Note: still orthogonal

Laplace System Analysis

Steps in Laplace system Analysis,

1. Laplace transform input signal
   - $X(s) = L\{x(t)\}$
2. Calculate system transfer function
   - $H(s)$
3. Calculate (Laplace) output using multiplication
   - $Y(s) = H(s)X(s)$
4. Inverse Laplace transform
   - $y(t) = L^{-1}\{Y(s)\}$

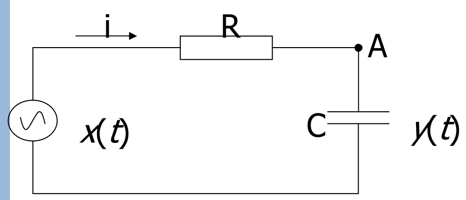
Key: Calculating system transfer function: $H(s)$

Laplace Transfer Function

- $H(s)$ completely defines the system
- Defined as:

$$H(s) = \frac{L\{y(t)\}}{L\{x(t)\}} = \frac{Y(s)}{X(s)}$$

Apply Kirchoff's current law at node A:



$$\frac{x(t) - y(t)}{R} = C\frac{dy}{dt}$$

Rearanging in terms of $x(t)$

$$\frac{x(t)}{R} = C\frac{dy}{dt} + \frac{y(t)}{R}$$

Laplace Transfer Function

Note: $L\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^+)$

Taking Laplace Transforms

$$\frac{X(s)}{R} = C\{sY(s) - y(0)\} + \frac{Y(s)}{R}$$

Assume zero initial conditions $y(0) = 0$

$$\frac{X(s)}{R} = \left\{Cs + \frac{1}{R}\right\} Y(s)$$

Transfer function (first order LPF):

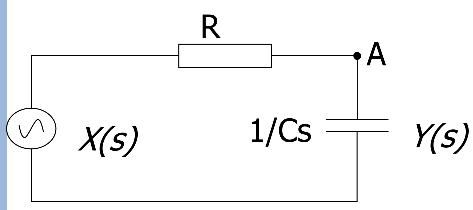
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(1 + RCs)}$$

Example 1.9 MGT

Note: similarity to H_n for FS

Laplace Circuit Analysis

- Transform circuit elements
  - then apply Kirchoff's current law
    - Impedance of Capacitor = 1/Cs
      - (Note: impedance of inductor = Ls)



$$\frac{X(s)-Y(s)}{R}=\frac{Y(s)}{1/Cs}=CsY(s)$$

$$H(s)=\frac{1}{(1+RCs)}$$

This is quick and easy way of doing circuit analysis

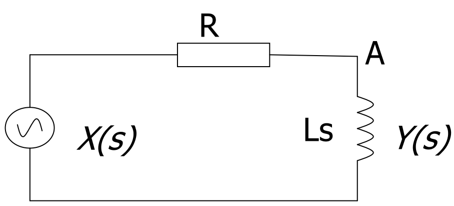
Laplace Circuit Analysis

Note: $L\left\{\int_0^t f(\xi)d\xi\right\}=\frac{F(s)}{s}$

- Impedance of Inductor = Ls

By Kirchoff's current law: $\left(\frac{x(t)-y(t)}{R}=\frac{1}{L}\int y(t)dt\right)$

$$\frac{X(s)-Y(s)}{R}=\frac{1}{sL}Y(s)$$



$$H(s)=\frac{1}{(1+\frac{R}{Ls})}=\frac{Ls}{(Ls+R)}$$

First order HPF

Summary

- Laplace transform similar to Fourier
  - Applicable to broad range of signals
  - Most one-sided (causal) finite energy and power
- Particularly useful for
  - solving ODE's i.e., analysing linear, time-invariant systems (e.g., circuit analysis)
- Based on summation (integration) of
  - (orthogonal) exponential (basis) functions
    - like Fourier series and transform
  - System response scaled versions of these
    - amplitude and phase change (only) at each frequency