LTI & Laplace Transforms

ELEC 3004: Signals, Systems & Control
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Lecture # 4 March 12, 2012

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Overview

- Laplace transform
  - Finite power signals
    1. Unilateral Laplace transform
    2. Bilateral Laplace transform
- Transform Analysis of Linear systems
  - Circuit Analysis
  - Transfer functions

Laplace Transform

- Problem: FT of a signal may not always exist!
  - finite power (and not periodic),
  - e.g., \( x(t) = u(t) \exp(-at) \) with \( a < 0 \)
  - Or \( x(t) = u(t) \cos(5t) \)
- Solution: Force signal to have finite energy
  - Multiply by convergence factor (\( \exp(-\sigma t) \))
  - i.e., new signal \( x_\sigma(t) = \exp(-\sigma t) x(t) \)
  - Therefore, FT of \( x_\sigma(t) \) exists

\[
X_\sigma(w) = \int_{-\infty}^{\infty} x_\sigma(t) \exp(-jwt) dt
\]

Rearranging...

\[
X_\sigma(w) = \int_{-\infty}^{\infty} x(t) \exp(-(\sigma + jw)t) dt
\]
Bilateral Laplace Transform

- For compactness we write
  \[ s = \sigma + j\omega \]

Note:

\[
X(s) = \int_{-\infty}^{\infty} x(t) \exp(-st) \, dt \quad \text{Laplace transform} \quad X(s) = L\{x(t)\}
\]

\[
x(t) = \frac{1}{2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \exp(st) \, ds \quad \text{Inverse Laplace transform} \quad x(t) = L^{-1}\{X(s)\}
\]

\[
x(t) = \exp(\sigma t) x_\sigma(t) \quad \text{i.e., by inverse of converge factor} \quad x(t) = \exp(\sigma t) \left( \int_{-\infty}^{\infty} X_\sigma(\omega) \exp(j\omega t) \, d\omega \right) \quad \text{i.e., } F^{-1}\{X_\sigma(\omega)\}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\sigma(\omega) \exp((\sigma + j\omega)t) \, d\omega \quad \text{Finally re-arrange}
\]

Unilateral Laplace Transform

- Problem with two-sided (bilateral) transform
  - choice of \( \sigma \) can cause ambiguities in \( L^{-1}\{X(s)\} \)
    - i.e., as different \( x(t) \)'s have same \( X(s) \)!
    - e.g., \( L\{\exp(at)u(t)\} = L\{-\exp(at)u(-t)\} \)

- Solution: assume \( x(t) \) to be causal (one-sided)
  - i.e., \( x(t) = 0 \) when \( t < 0 \)
  - This is termed the unilateral Laplace Transform
  - Integration is now from \( 0 \leq t < \infty \)

- This solution works for most practical signals
  - Bilateral required for (non-deterministic) random signals
    - (see later)
Unilateral Laplace Transform

• One-sided Laplace transform

\[ X(s) = \int_{0^-}^{\infty} x(t) \exp(-st) \, dt \]

0\(^{-}\) indicates origin is included in integration \(0 \leq t < \infty\)

• Laplace transform
  – \(X(s) = L\{x(t)\}\)

• Inverse Laplace transform
  – \(x(t) = L^{-1}\{X(s)\}\)

Convergence of Laplace Transform

• Consider signal

\[ x(t) = \exp(-at)u(t) \]

\[ X(s) = \int_{0}^{\infty} \exp(-(s+a)t) \, dt \]

\[ = \int_{0}^{\infty} \exp(-(\sigma+a)t) \exp(-jwt) \, dt \]

\[ = \frac{-1}{s+a} \left[ \exp(-(s+a)t) \right]_{0}^{\infty} \]

\[ = \frac{1}{(\sigma+a) + jw}, \quad \sigma + a > 0 \]

\[ X(s) = \frac{1}{s+a}, \quad \Re\{s\} > -a \]

Effectively same as \(\leftrightarrow\) Fourier Transform

• Convergence dependent on both \(\sigma\) and \(w\)
  – Note: \(\Re\{s\} = \sigma\)

• Region of Convergence (ROC)
  – Finite integral (energy)
Laplace Examples

Unit step function: 
\[ L\{u(t)\} = \int_0^\infty u(t) \exp(-st) \, dt \]
\[ = \int_0^\infty \exp(-st) \, dt \]
\[ = \left[ -\frac{\exp(-st)}{s} \right]_0^\infty = \frac{1}{s}, \quad \sigma > 0 \]

Impulse function: 
\[ L\{\delta(t)\} = \int_0^\infty \delta(t) \exp(-st) \, dt \]
\[ = \int_0^\infty \delta(t) \, dt \]

Remember:
\[ \int_0^\infty f(t) \delta(t) \, dt = f(0) \]
\[ \int_0^\infty \delta(t) \, dt = 1 \]

Interpretation of Laplace Transform

- Represents signals, \( x(t) \), as sum of
  - growing/decaying cosine waves
- at all frequencies (continuous), \( X(s) \)
  - \( \exp(\sigma t) |X(s)| \, dw/2\pi \) is amplitude of growing/decaying cosine wave
    - In frequency band \( w \) to \( w + dw \)
  - \( \angle X(s) \) is phase shift of cosine wave
- parameter \( \sigma (\Re\{s\}) \) determines rate of growth or decay
  - Note: \( \sigma = 0 \) is the Fourier Transform ☺
    - Constant amplitude cosine waves
Complex Phasors

As per Fourier Transform

\[ A \exp(st) = A \exp(\sigma t) \cos(\omega t) + jA \exp(\sigma t) \sin(\omega t) \]

Decaying \( \sigma < 0 \)

Growing \( \sigma > 0 \)

Linear Transforms

- So far, we have looked at
  - Fourier series
  - Trigonometrical & Complex
  - Fourier transform
  - Laplace transform
- All represent signals as a
  - Weighted sum (or integration) of
  - Complex exponentials (that are orthogonal)
  - e.g., complex FS, \( x(t) = \sum X_n \exp(\omega_n t) \)
- This relates directly to linear systems
Linear Transforms & Linear Systems

Useful, due to two properties of linear systems

1. Superposition principle
   - Inputs are complex exponentials (sinusoids)
   - Output is same exponentials, but with different weights, i.e., delay and amplify/attenuate input

\[ H \{ax_1(t) + bx_2(t)\} = aH \{x_1(t)\} + bH \{x_2(t)\} \]

In other words, each phasor can be considered individually and output calculated by summation of the individual phasors

2. Ordinary differential equations (ODE)
   - Output \(y(t)\) related to input \(x(t)\) by ODE

\[ a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m} \]

Differentiation is simple:
- Frequency unchanged
- Magnitude changes

Note: initially consider harmonically related sinusoids \(n\omega_0\) as per Fourier Series
Using Kirchhoff’s laws, system described (modelled) by ODE
(1 energy store = 1st order)

\[ a_0 y(t) + a_1 \frac{dy}{dt} = b_0 \exp(jna_0 t) \]
\[ y(t) + RC \frac{dy}{dt} = \exp(jna_0 t) \]

Specifically:
- \( a_0 = 1 \)
- \( a_1 = RC \)
- \( b_0 = 1 \)

derivation shortly...

Using Kirchhoff’s laws, system described (modelled) by ODE
(1 energy store = 1st order)

\[ x(t) \text{ sinusoidal input} \]

\[ y(t) \text{ (output)} \]

\[ R \]

\[ C \]

**Example RC Circuit**

Note: \[ \frac{dy_n(t)}{dx} = (jna_0) y_n(t) \]

- If input single sinusoid \( x_n(t) = \exp(jn\omega_0 t) \)
  - and system linear, (steady-state) output has form:

\[ y_n(t) = K \exp(jna_0 t) \]

where \( K \) is complex constant (weight)

\[ a_0 y_n(t) + a_1 (jna_0) y_n(t) = b_0 \exp(jna_0 t) \]

\[ y_n(t) = \frac{b_0}{a_0 + (jna_0)a_i} \exp(jna_0 t) \]

\[ y_n(t) = H_n \exp(jna_0 t) \]

\[ H_n = \frac{b_0}{a_0 + (jna_0)a_i} = \frac{1}{1+(jna_0)RC} \]

Where \( H_n \) is the system **transfer function** to this single phasor (K)
Response of Linear System

- Input represented as Fourier series

\[ X_1 \exp(jw_0 t) \]
\[ X_2 \exp(j2w_0 t) \]
\[ \vdots \]
\[ X_n \exp(jnw_0 t) \]

\[ \Sigma \]

Linear System

\[ y(t) \]

Each phasor is orthogonal and \( \therefore \) can be evaluated separately

- Applying superposition principle
  - Applying individual phasors \& shifting \( X_n \) terms

\[ \exp(jw_0 t) \]
\[ \exp(j2w_0 t) \]
\[ \vdots \]
\[ \exp(jnw_0 t) \]

Linear System

\[ H_1 \exp(jw_0 t) \]
\[ H_2 \exp(j2w_0 t) \]
\[ \vdots \]
\[ H_n \exp(jnw_0 t) \]

\[ \Sigma \]

Linear System

\[ y(t) \]
Response of Linear System

- Output represented as Fourier series
  - output FS coefficients $Y_n = H_n X_n$

\[
y(t) = \sum_{n=-\infty}^{\infty} H_n X_n \exp(jn\omega_0 t)
\]

General Approach

- Represent input as weighted sum of
  - Complex exponential basis functions,
    - e.g., FS: $\exp(jn\omega_0 t)$: sinusoids at harmonic frequencies
- Basis functions are orthogonal
  - Amplitude (and phase) at each freq. evaluated separately
- System described by ODE
  - Response to exponentials (differentials) easy to calculate
    - Frequency remains constant, Amp. & Phase change
      - e.g., if response 2nd harmonic is $2\omega_0 \exp(2j\omega_0 t)$
  - System is linear
    - Output is sum of responses of individual exponentials
      - i.e., sum response at each frequency
Laplace Transforms

- General approach was illustrated with
  - Periodic input (i.e., we used Fourier series)
  - Steady state response
- But, in general interested in
  - Non-periodic input and
  - Both transient & steady state response
- So, we use the Laplace transform
  - Basis functions exp(\(st\)), where \(s = \sigma + jw\)
  - Output response is \(H(s) \exp(st)\)
  - Where \(H(s)\) is system transfer function

Note: still orthogonal

Laplace System Analysis

Steps in Laplace system Analysis,
1. Laplace transform input signal
   - \(X(s) = L\{x(t)\}\)
2. Calculate system transfer function
   - \(H(s)\)
3. Calculate (Laplace) output using multiplication
   - \(Y(s) = H(s)X(s)\)
4. Inverse Laplace transform
   - \(y(t) = L^{-1}\{Y(s)\}\)

Key: Calculating system transfer function: \(H(s)\)
Laplace Transfer Function

• $H(s)$ completely defines the system
• Defined as:

$$H(s) = \frac{L\{y(t)\}}{L\{x(t)\}} = \frac{Y(s)}{X(s)}$$

Apply Kirchoff's current law at node A:

$$x(t) - y(t) = C \frac{dy}{dt}$$

Rearranging in terms of $x(t)$

$$\frac{x(t)}{R} = C \frac{dy}{dt} + \frac{y(t)}{R}$$

Example 1.9

Taking Laplace Transforms

$$\frac{X(s)}{R} = C\{sY(s) - y(0)\} + \frac{Y(s)}{R}$$

Assume zero initial conditions $y(0) = 0$

$$\frac{X(s)}{R} = \left\{ Cs + \frac{1}{R} \right\} Y(s)$$

Transfer function (first order LPF):

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1 + RCS}$$

Note: similarity to $H_n$ for FS
Laplace Circuit Analysis

- Transform circuit elements
  - then apply Kirchoff's current law
- Impedance of Capacitor = 1/Cs
  - (Note: impedance of inductor = Ls)

\[
\begin{align*}
X(s) - Y(s) &= Y(s) \frac{s}{1/Cs} = CsY(s) \\
R H(s) &= \frac{1}{1 + RCs}
\end{align*}
\]

This is quick and easy way of doing circuit analysis

Laplace Circuit Analysis

- Impedance of Inductor = Ls

By Kirchoff's current law:

\[
\begin{align*}
\frac{x(t) - y(t)}{R} &= \frac{1}{L} \int y(t) dt \\
X(s) - Y(s) &= \frac{1}{sL} Y(s) \\
H(s) &= \frac{1}{\left(1 + \frac{R}{Ls}\right)} = \frac{Ls}{(Ls + R)}
\end{align*}
\]

First order HPF
Summary

• Laplace transform similar to Fourier
  - Applicable to broad range of signals
  - Most one-sided (causal) finite energy and power
• Particularly useful for
  - solving ODE’s i.e., analysing linear, time-invariant systems (e.g., circuit analysis)
• Based on summation (integration) of
  - (orthogonal) exponential (basis) functions
    • like Fourier series and transform
  - System response scaled versions of these
    • amplitude and phase change (only) at each frequency