Sampling

ELEC 3004: Signals, Systems & Control
Dr. Surya Singh, Prof. Brian Lovell & Dr. Paul Pounds
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elec3004@itee.uq.edu.au
http://courses.itee.uq.edu.au/elec3004/2012s1/

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Announcements

• Practicals next week!
Therefore, No Tutorials next week

• Practical 1: Pre-lab will be posted this evening!

• No Pre-Lab == No Lab Admission

• Please go to the lab you have been assigned to!

Introduction

• Most naturally occurring signals are continuous-valued
  — continuous-time (CT) and continuous-value (CV)

• Analogue system, e.g., analogue filter
  — problems: component tolerances, variation with temperature and age
  — limited to time invariant systems (non-adaptive)

• Digital System (DSP: software, VLSI, or FPGA)
  — can replace analogue systems, more robust
  — can also implement adaptive and non-linear fctns
    • still some issues: quantisation, aliasing (later)
**Analogue & Digital Systems**

**Analogue System**

\[ x(t) \rightarrow \text{Analogue Filter} \rightarrow y(t) \]

**Digital System**

\[ x(t) \rightarrow \text{sampling} \rightarrow \text{quantisation} \rightarrow x[n] \rightarrow \text{DSP} \rightarrow y[n] \rightarrow \text{reconstruction} \rightarrow y(t) \]

**Key:** CT = continuous-time, CV = continuous-valued, DT = discrete-time, DV = discrete valued

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**Mathematics of Sampling and Reconstruction**

\[ x(t) \rightarrow x_c(t) \rightarrow \text{DSP} \rightarrow \text{reconstruction} \rightarrow y(t) \]

**Impulse train**

\[ \delta_T(t) = \sum \delta(t - n\Delta t) \]

**Impulse train**

\[ \delta_T(t) = \sum \delta(t - n\Delta t) \]

**Sampling frequency**

\[ f_s = 1/\Delta t \]

**Gain**

\[ \text{Gain} \]

**Cut-off frequency**

\[ f_c \]

**Freq**

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**Mathematical Model of Sampling**

- $x(t)$ multiplied by impulse train $\delta_f(t)$

\[
X_c(t) = x(t)\delta_f(t) = x(t)[\delta(t) + \delta(t-\Delta t) + \delta(t-2\Delta t) + \cdots] = \sum_{n} x(n\Delta t)\delta(t-n\Delta t)
\]

- $x_c(t)$ is a train of impulses of height $x(t)|_{t=n\Delta t}$
Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes $x_c(t)$ to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time = convolution in frequency
  - $F\{x(t)\} = X(w)$
  - $F\{\delta_T(t)\} = \sum \delta(w - 2\pi n/\Delta t)$
  - i.e., an impulse train in the frequency domain

In the frequency domain we have

$$X_c(w) = \frac{1}{2\pi} \left( X(w) \ast \frac{2\pi}{\Delta t} \sum_n \delta \left( w - \frac{2\pi n}{\Delta t} \right) \right)$$

$$= \frac{1}{\Delta t} \sum_n X\left( w - \frac{2\pi n}{\Delta t} \right)$$

- Let’s look at an example
  - where $X(w)$ is triangular function
  - with maximum frequency $w_m$ rad/s
  - being sampled by an impulse train, of frequency $w_s$ rad/s

Remember convolution with an impulse? Same idea for an impulse train
Fourier transform of original signal $X(\omega)$ (signal spectrum)

Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)

Fourier transform of sampled signal

Original spectrum convolved with spectrum of impulse train

Spectrum of sampled signal

Reconstruction filter (ideal lowpass filter)

Spectrum of reconstructed signal

$X(\omega) = H_L(\omega) X_c(\omega)$
Sampling Frequency

- In this example it was possible to recover the original signal from the discrete-time samples.
- But is this always the case?
- Consider an example where the sampling frequency $\omega_s$ is reduced:
  - i.e., $\Delta t$ is increased.

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**Original Spectrum**

- Fourier transform of impulse train (sampling signal)
- Amplitude spectrum of sampled signal

Replica spectrums overlap with original (and each other)
This is **Aliasing**
Amplitude spectrum of sampled signal

... Original Replica 1 Replica 2 ...

Reconstruction filter (ideal lowpass filter)

Due to overlapping replicas (aliasing) the reconstruction filter cannot recover the original spectrum

Sampling Theorem

- The Nyquist criterion states:
  
  To prevent aliasing, a **bandlimited** signal of bandwidth $w_B$ rad/s must be sampled at a rate greater than $2w_B$ rad/s

  $$-w_s > 2w_B$$

  Note: this is a $>$ sign not a $\geq$

  Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter
Time Domain Analysis of Sampling

- Frequency domain analysis of sampling is very useful to understand:
  - sampling \((X(w)\sum \delta(w - 2\pi n/\Delta t))\)
  - reconstruction (lowpass filter removes replicas)
  - aliasing (if \(w_r \leq 2w_B\))
- Time domain analysis can also illustrate the concepts:
  - sampling a sinewave of increasing frequency
  - sampling images of a rotating wheel

A signal of the original frequency is reconstructed
A signal with a reduced frequency is recovered, i.e., the signal is aliased to a lower frequency (we recover a replica).
Nyquist is not enough …

1Hz Sin Wave: Sin(2πt) → 2 Hz Sampling

Time(s) Normalized magnitude

1Hz Sin Wave: Sin(2πt) → 4 Hz Sampling

Time(s) Normalized magnitude
Sampled Spectrum $w_s > 2w_m$

Original and replica spectrums overlap

Original freq recovered

Sampled Spectrum $w_s < 2w_m$

Lower frequency recovered ($w_s - w_m$)

Temporal Aliasing

90° clockwise rotation/frame
Clockwise rotation perceived

270° clockwise rotation/frame
(90°) anticlockwise rotation perceived i.e., aliasing

Require LPF to ‘blur’ motion
Time Domain Analysis of Reconstruction

- Frequency domain: multiply by ideal LPF
  - ideal LPF: ‘rect’ function (gain $\Delta t$, cut off $w_c$)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with ‘sinc’ function
  - as $F^{-1}\{\Delta t \text{rect}(w/w_c)\} = \Delta t w_c \text{sinc}(w_c ft)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \text{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$
Sampling and Reconstruction
Theory and Practice

- Signal is bandlimited to bandwidth $W_f$
  - Problem: real signals are not bandlimited
    - Therefore, require (non-ideal) anti-aliasing filter
- Signal multiplied by ideal impulse train
  - Problems: sample pulses have finite width
  - and not $\otimes$ in practice, but sample & hold circuit
- Samples discrete-time, continuous valued
  - Problem: require discrete values for DSP
    - Therefore, require A/D converter (quantisation)
- Ideal lowpass reconstruction (‘sinc’ interpolation)
  - Problems: ideal lowpass filter not available
    - Therefore, use D/A converter and practical lowpass filter

Practical DSP System

\[ w_c < w_s/2 \]
\[ w_c = w_s/2 \]
\[ w_s > 2w_f \]

Note:
- $w_c > 2w_f$
- $w_c < w_s/2$
- $w_c = w_s/2$
- $w_s > 2w_f$

- Sampling and Hold
- A/D Converter
- DSP Processor
- D/A Converter
- DSP Board
Practical Anti-aliasing Filter

- Non-ideal filter
  - $\omega_c = \omega_s / 2$
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies $> \omega_c$ may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
  - e.g., speech naturally bandlimited (say < 8KHz)
  - Natural signals have a (approx) 1/f spectrum
  - so, in practice aliasing is not (usually) a problem

Finite Width Sampling

- Impulse train sampling not realisable
  - sample pulses have finite width (say nanosecs)
This produces two effects,
1. Impulse train has sinc envelope in frequency domain
   - impulse train is square wave with small duty cycle
   - Reduces amplitude of replica spectrums
     - smaller replicas to remove with reconstruction filter ☺
2. Averaging of signal during sample time
   - effective low pass filter of original signal
     - can reduce aliasing, but can reduce fidelity ☺
     - negligible with most S/H ☺
Amplitude spectrum of original signal

Fourier transform of sampling signal (pulses have finite width)

sinc envelope
Zero at harmonics
1/duty cycle

Fourier transform of sampled signal

Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every $\Delta t$ seconds
  2. holds that value constant until next sample
- Produces ‘staircase’ waveform, $x(n\Delta t)$
Quantisation

- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x_q[n] = q(x(n\Delta t))$, where $q()$ is non-linear rounding fctn
    - Output modeled as $x_q[n] = x(n\Delta t) + e[n]$
- Approximation process
  - Therefore, loss of information (unrecoverable)
  - Known as 'quantisation noise' ($e[n]$)
  - Error reduced as number of bits in A/D increased
    - i.e., $\Delta x$ quantisation step size reduces

$$|e[n]| \leq \frac{\Delta x}{2}$$

Input-output for 4-bit quantiser (two’s compliment)

$$\Delta x = \frac{2A}{2^m - 1}$$
where $A = \text{max amplitude}$
$m = \text{no. quantisation bits}$
Signal to Quantisation Noise

- To estimate SQNR we assume
  - $\varepsilon$ is uncorrelated to signal and a uniform random process
- assumptions not always correct!
  - not the only assumptions we could make...
- Also known a ‘Dynamic range’ ($R_D$)
  - expressed in decibels (dB)
  - ratio of power of largest signal to smallest (noise)

$$R_D = 10\log_{10}\left(\frac{P_{\text{signal}}}{P_{\text{noise}}}\right)$$
Dynamic Range

Need to estimate:

1. Noise power
   - uniform random process: \( P_{\text{noise}} = \Delta x^2 / 12 \)

2. Signal power
   - (at least) two possible assumptions
     1. sinusoidal: \( P_{\text{signal}} = A^2 / 2 \)
     2. zero mean Gaussian process: \( P_{\text{signal}} = \sigma^2 \)
        - Note: as \( \sigma = A/3 \): \( P_{\text{signal}} = A^2 / 9 \)
        - where \( \sigma^2 \) = variance, \( A \) = signal amplitude

Regardless of assumptions: \( R_D \) increases by 6dB for every bit that is added to the quantiser

1 extra bit halves \( \Delta x \)
i.e., \( 20 \log_{10}(1/2) = 6 \text{dB} \)
Practical Reconstruction

Two stage process:
1. Digital to analogue converter (D/A)
   - zero order hold filter
   - produces 'staircase' analogue output
2. Reconstruction filter
   - non-ideal filter: \( w_c = w_s / 2 \)
   - further reduces replica spectrums
   - usually 4th – 6th order e.g., Butterworth
     • for acceptable phase response

D/A Converter

• Analogue output \( y(t) \) is
  - convolution of output samples \( y(n\Delta t) \) with \( h_{ZOH}(t) \)

\[
\begin{align*}
  y(t) &= \sum_n y(n\Delta t)h_{ZOH}(t - n\Delta t) \\
  h_{ZOH}(t) &= \begin{cases} 
  1, & 0 \leq t < \Delta t \\
  0, & \text{otherwise}
  \end{cases} \\
  H_{ZOH}(w) &= \Delta t \exp\left( -jw\Delta t \right) \sin(w\Delta t / 2) / w\Delta t / 2
\end{align*}
\]

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required
Zero Order Hold (ZOH)

- **ZOH impulse response**
  - $h_{ZOH}(t)$
  - $0 \leq t \leq \Delta f$

- **ZOH amplitude response**
  - $|R_{in}(\theta)|$
  - $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi$

- **ZOH phase response**
  - $\angle h_{ZOH}(\theta)$

'**staircase**' output from D/A converter (ZOH)

- **Time (sec)**
  - 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

- **Amplitude (V)**
  - 0, 2, 4, 6, 8, 10, 12, 14, 16

- **Output samples**
- **D/A output**
Smooth output from reconstruction filter

- D/A output
- Reconstruction filter output

Original Signal -> After Anti-aliasing LPF -> After Sample & Hold

- After Reconstruction LPF
- After D/A
- After A/D

Complete practical DSP system signals
Summary

• Theoretical model of Sampling
  – bandlimited signal ($f_b$)
  – multiplication by ideal impulse train ($f_s > 2f_b$)
    • convolution of frequency spectrums (creates replicas)
  – Ideal lowpass filter to remove replica spectrums
    • $f_c = f_s/2$
    • Sinc interpolation

• Practical systems
  – Anti-aliasing filter ($f_c < f_s/2$)
  – A/D (S/H and quantisation)
  – D/A (ZOH)
  – Reconstruction filter ($f_c = f_s/2$)

Questions

1. A 7 kHz sine wave is sampled at 10 kHz. What frequencies are present at the output of the A2D?
2. Determine the voltage and power ratios of a pair of sinusoidal signals that have a 3dB ratio.
3. A 1 kHz ±5 V sine wave is applied to a 14-bit A2D operating from a ±10 V supply sampling at 10 kHz. What is the SQNR of the A2D in dB?
Questions

1. $\pm 7, 10 - 7 = \pm 3, 10 + 7 = \pm 17, 20 - 7 = \pm 13, \ldots$
   - Note aliasing and symmetry of spectrum
2. Rearrange: $3 = 10 \log_{10} \left( \frac{P_1}{P_2} \right) = 10^{3/10} = 2$
   - Rearrange: $3 = 20 \log_{10} \left( \frac{V_1}{V_2} \right) = 10^{3/20} = \sqrt{2}$
3. A2D resolution: $\Delta x = (2 \times 10) / (2^{14} - 1)$
   - $P_{\text{noise}} = \Delta x^2 / 12$
   - $P_{\text{signal}} = 5^2 / 2$

$$SQNR = 10 \log_{10} \left( \frac{P_{\text{signal}}}{P_{\text{noise}}} \right) dB$$