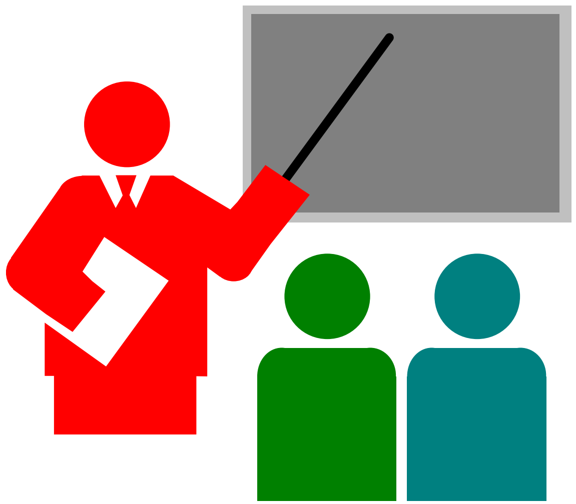
Recursive or IIR Filters

And why they should be avoided

Recursive Filters

- $x(n) \Rightarrow X(z)$
- $x(n-k) \Rightarrow z^{-k}X(z)$
- Why z^{-1}?
- z^{-1} corresponds to a unit delay; the basic building block.
- Zeros: Thumbtacks to hold surface down
- Poles: Builds mountains



Different Filter Models

All Zero (FIR)
MA

$$X(z) = a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1}$$

Pole Zero (IIR)
ARMA

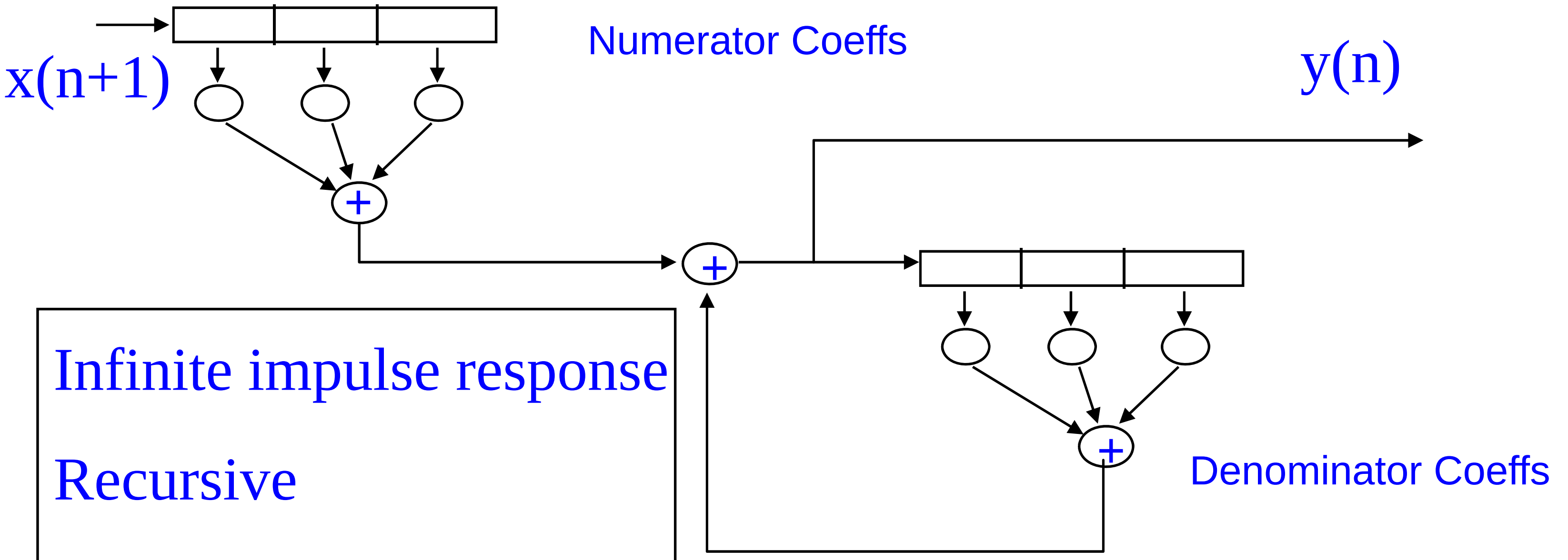
$$Y(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1}}{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1}}$$

All Pole (IIR)
AR

$$Z(z) = \frac{A}{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1}}$$

IIR Filter Structures

- The Z transform has an infinite number of terms (denominator polynomial), so there must be some feedback in the filter.



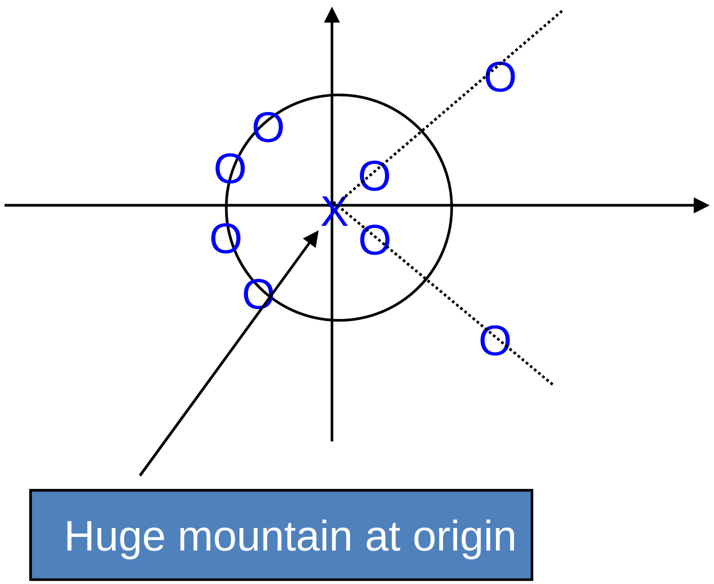
Infinite impulse response

Recursive

Pole-Zero

DIRECT IMPLEMENTATION

Linear Phase Filter



Zeros on unit circle in conj pairs

Zeros off unit circle in reciprocal
conj pairs

$P(z) = z^{M} P(z^{-1})$

Reversal of Coefficients

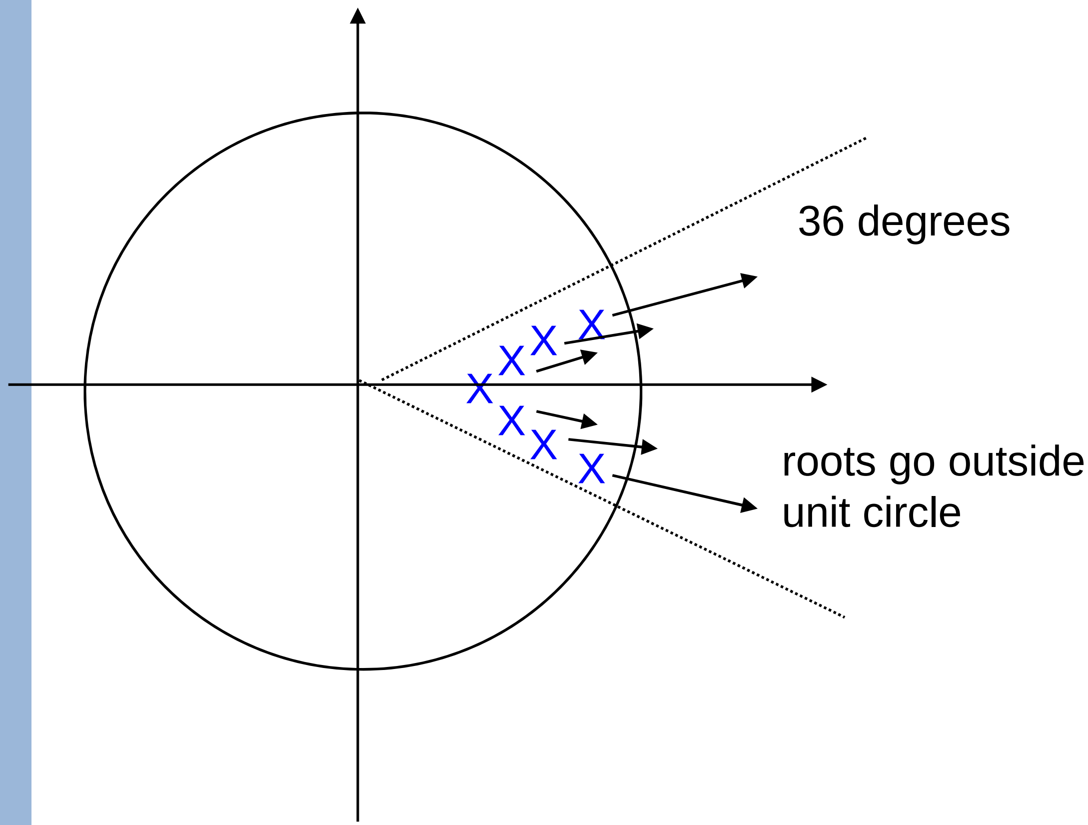
Implementation of IIR Filters

If you use direct implementation
and the number of poles >5
Bandwidth/Fs <0.1
16 bit arithmetic

↓

Then you have built an oscillator

Why?



Filters always oscillate in the low order bits.

Roots spread out like point charges due to finite precision

Works fine in Matlab (80 bit precision) but doesn't work in practice.

Why?

$$P(z) = \prod_i (z - z_i)$$

$$= \sum a_k z^{N-k}$$

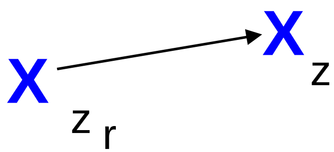
$$= a_0 z^N + a_1 z^{N-1} + \cdots + a_N$$

Replace a_l with $a_l + \Delta a_l$

$$\hat{P}(z) = P(z) + \Delta a_l z^{N-l}$$

Need roots $\Rightarrow \prod_i (z - z_i) + \Delta a_l z^{N-l} = 0$

Pick one root



$$(z - z_r)\prod^{i\neq r}(z - z_i) + \Delta a_l z^{N-l} = 0$$

$$z = z_r - \frac{\Delta a_l z^{N-l}}{\prod^{i\neq r}(z - z_i)}$$

Guess $z = z_r$, solve for z, keep substituting

$$z \approx z_r - \frac{\Delta a_l z_r^{N-l}}{\prod^{i\neq r}(z_r - z_i)}$$

Sensitivity

Sensitivity

$$\frac{1}{\prod^{i \neq r}(z_r - z_i)}$$

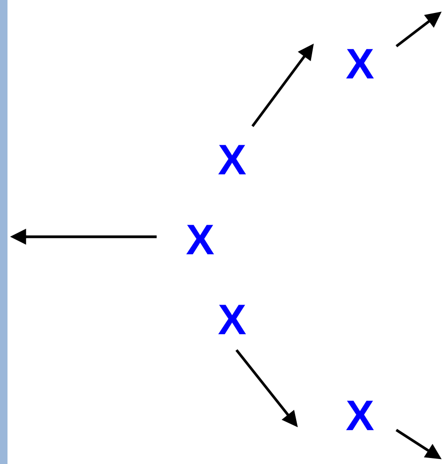
If distance = 0.1

Then reciprocal = 10000

Increase sampling by 10 (small bandwidth)

Then reciprocal = 10^{8}

Roots always spread out like point charges

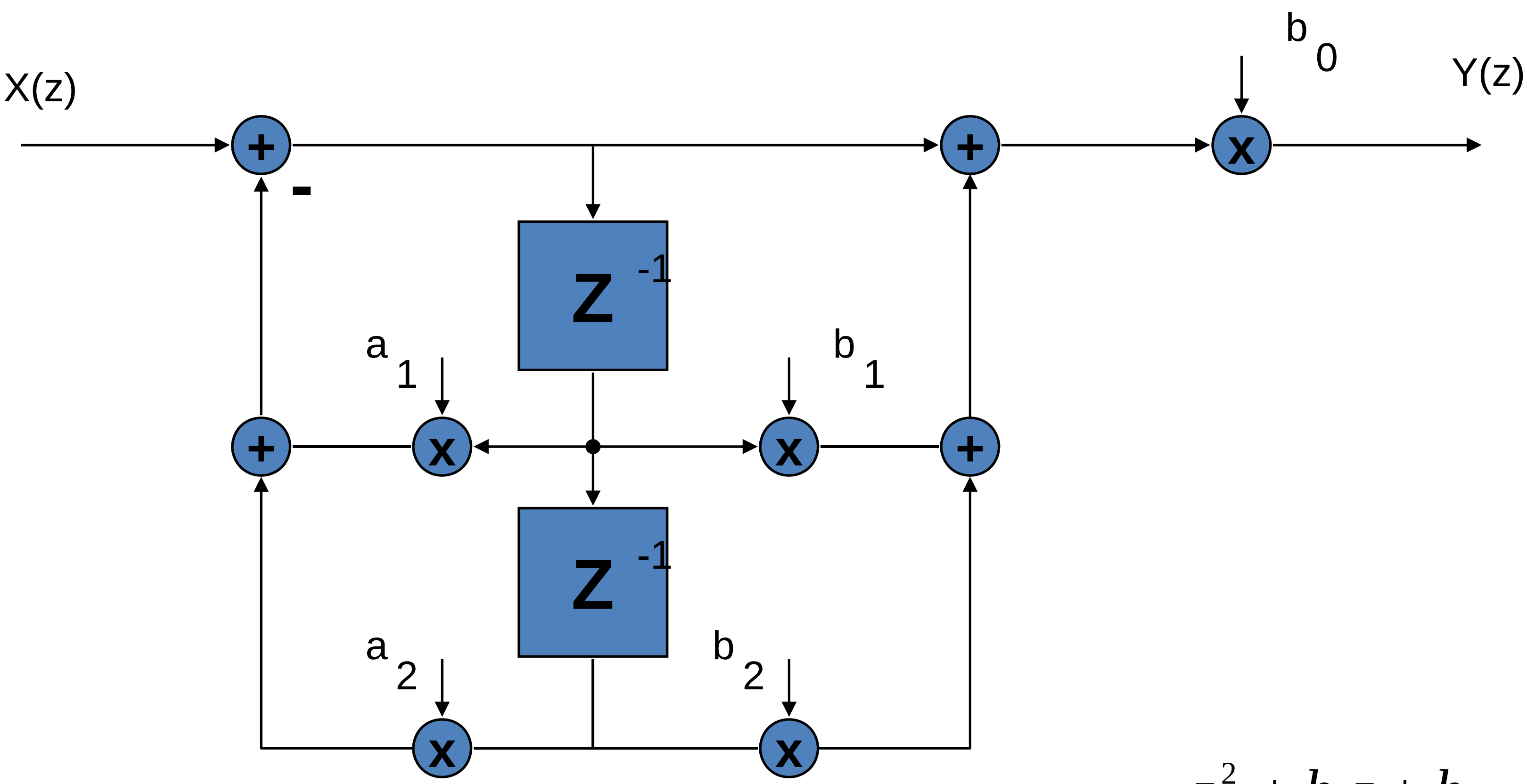


Solution

- Don't use IIR filters. Stick with polyphase FIR.
  - similar computational efficiency, always stable, low sensitivity to coefficient quantization, linear (any) phase
- Use cascade of low (1st or 2nd) order polynomials
  - similar to analog active filter design
- Alternate structures may reduce stability problems
  - Direct Form (very poor performance)
  - First Canonic Form
  - Second Canonic Form
  - Parallel Form
  - Cascade Form

See Oppenheim & Schafer

2nd Order Canonical Form I



$$H(z) = b_0 \frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$$

Canonical Form I

$$H(z) = b_0 \frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$$

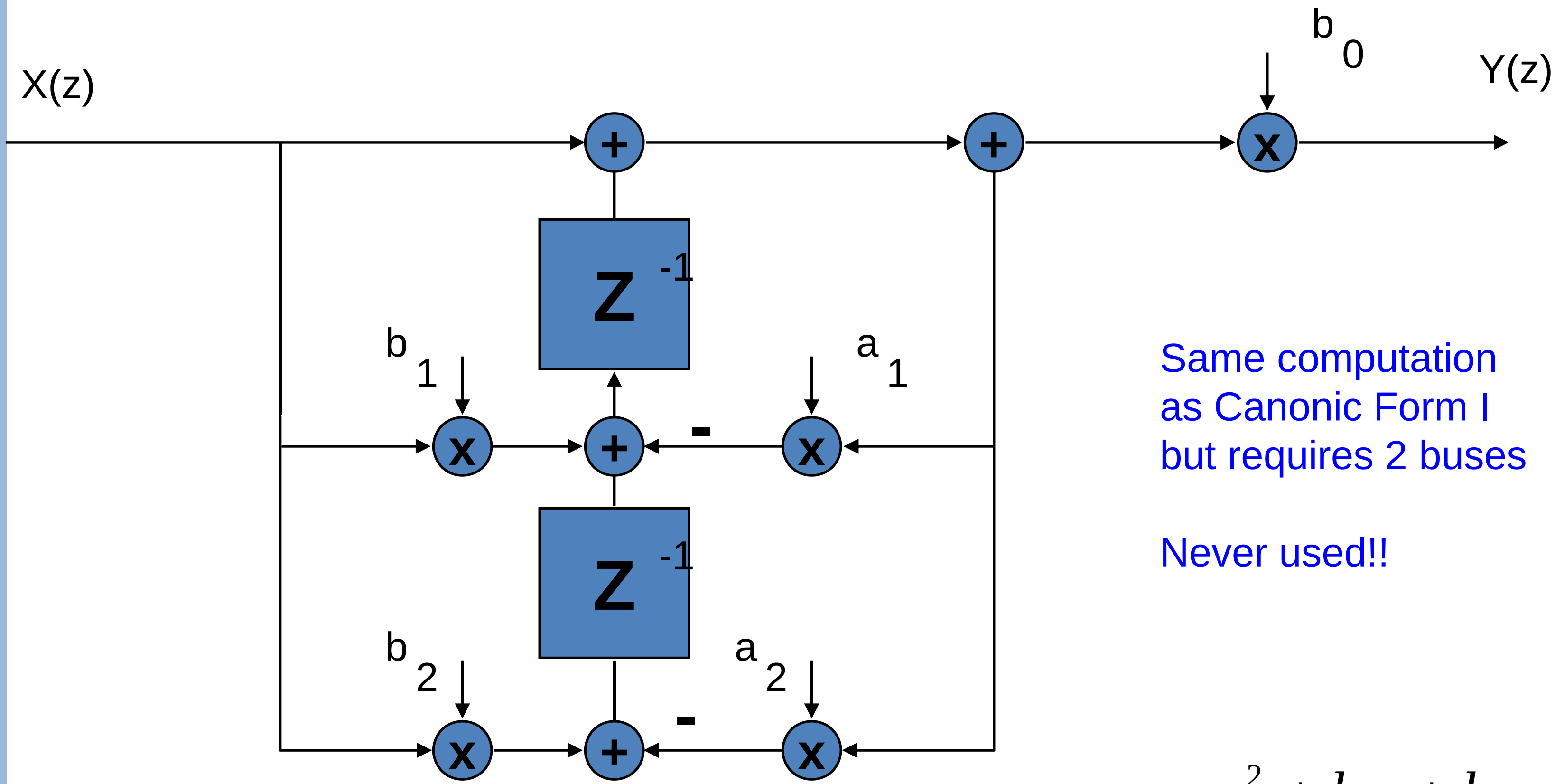
$$\frac{KY(z)}{X(z)} = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$\frac{KY(z)}{X(z)} = \frac{W(z)}{X(z)} \frac{KY(z)}{W(z)}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$\frac{KY(z)}{W(z)} = 1 + b_1 z^{-1} + b_2 z^{-2}$$

Canonical Form II



Same computation
as Canonic Form I
but requires 2 buses

Never used!!

$$H(z) = b_0 \frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$$

Pole Sensitivity

$$T(z) = \frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$$

$$z^2 + a_1 z + a_2$$

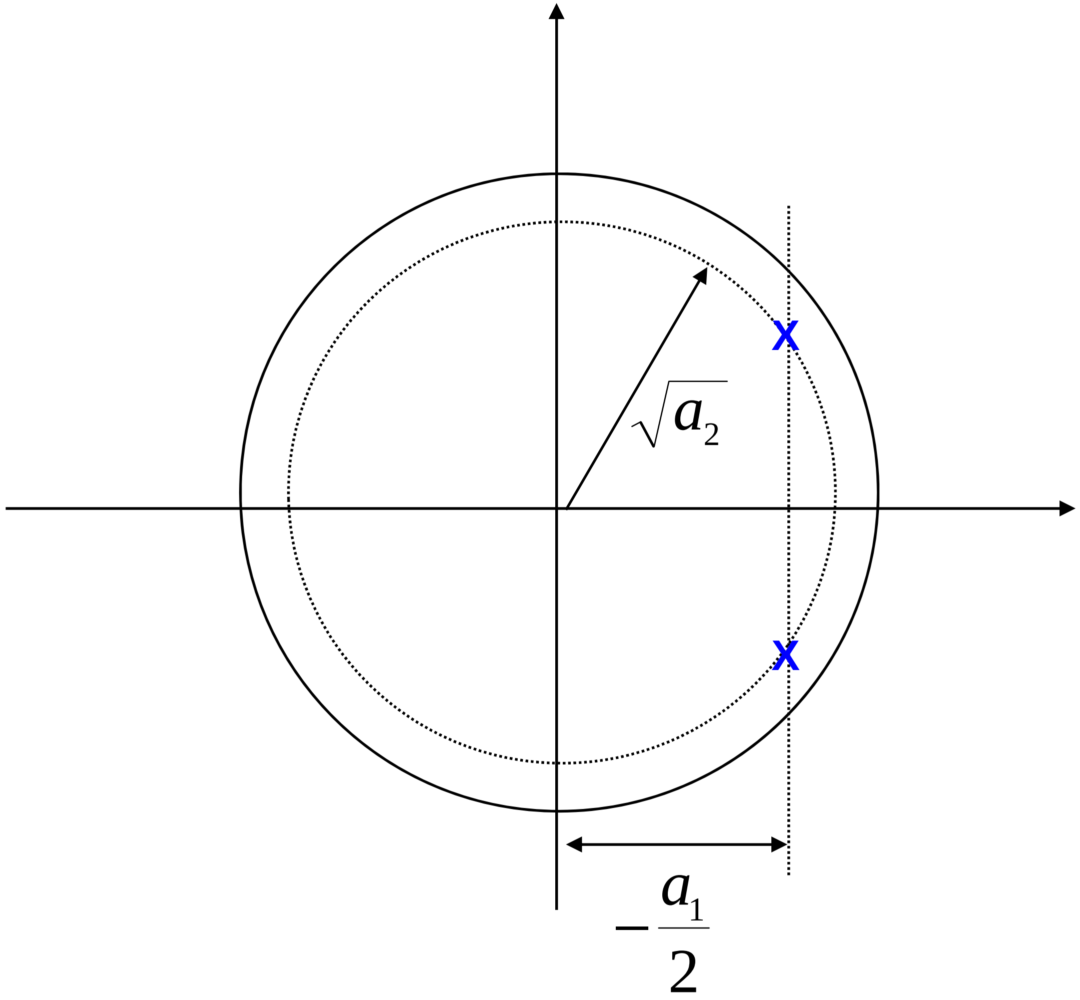
-2 Real Part

Magnitude Squared

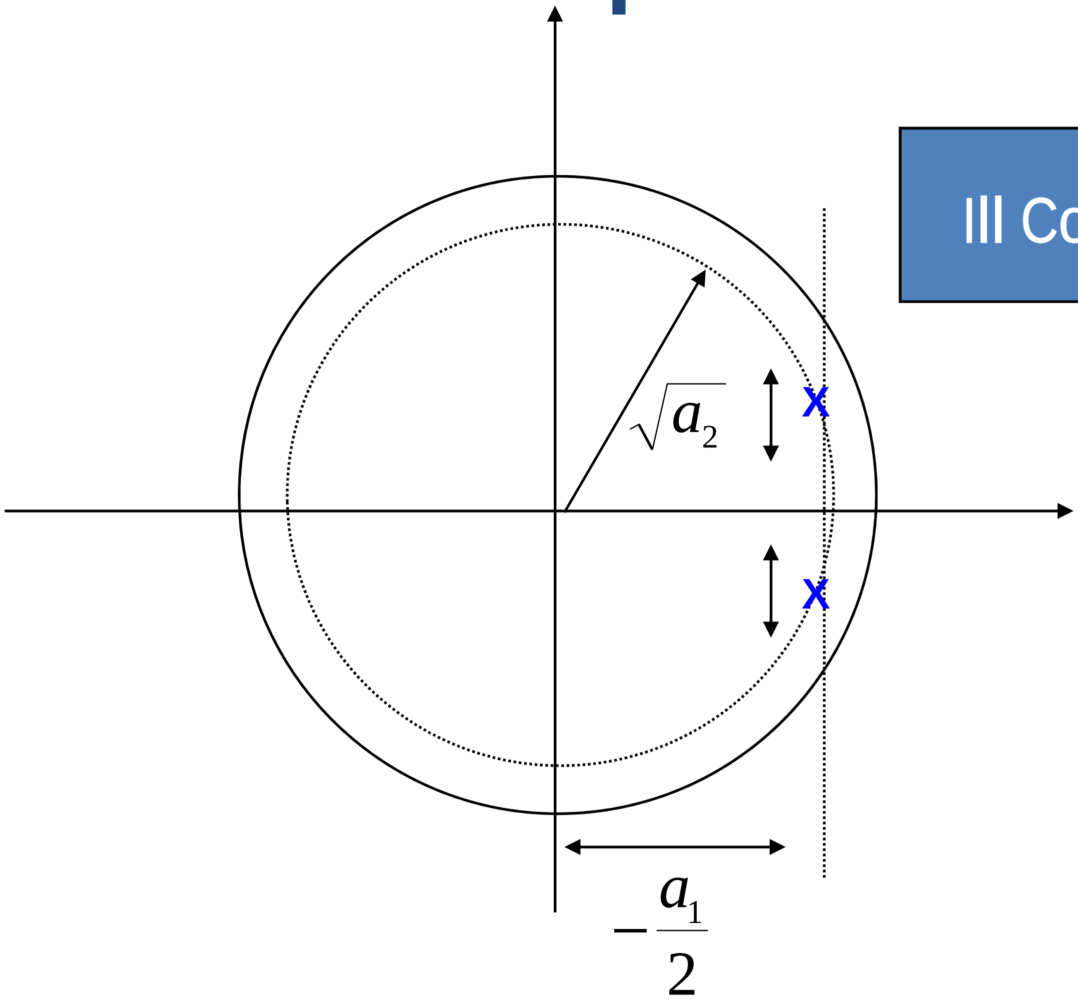
$$e.g. \quad z^2 - 1.9z - 0.98$$

$$\text{roots at} \quad \mathrm{Re} = 0.95, \mathrm{Mag} = 0.99$$

Pole Sensitivity



Increase Sample Rate



Ill Conditioned

Why?

$$T(z) = \frac{Y(z)}{X(z)} = \frac{z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2}$$

$$Y(z)(z^2 + a_1 z + a_2) = X(z)(z^2 + b_1 z + b_2)$$

$$Y(z)(1 + a_1 z^{-1} + a_2 z^{-2}) = X(z)(1 + b_1 z^{-1} + b_2 z^{-2})$$

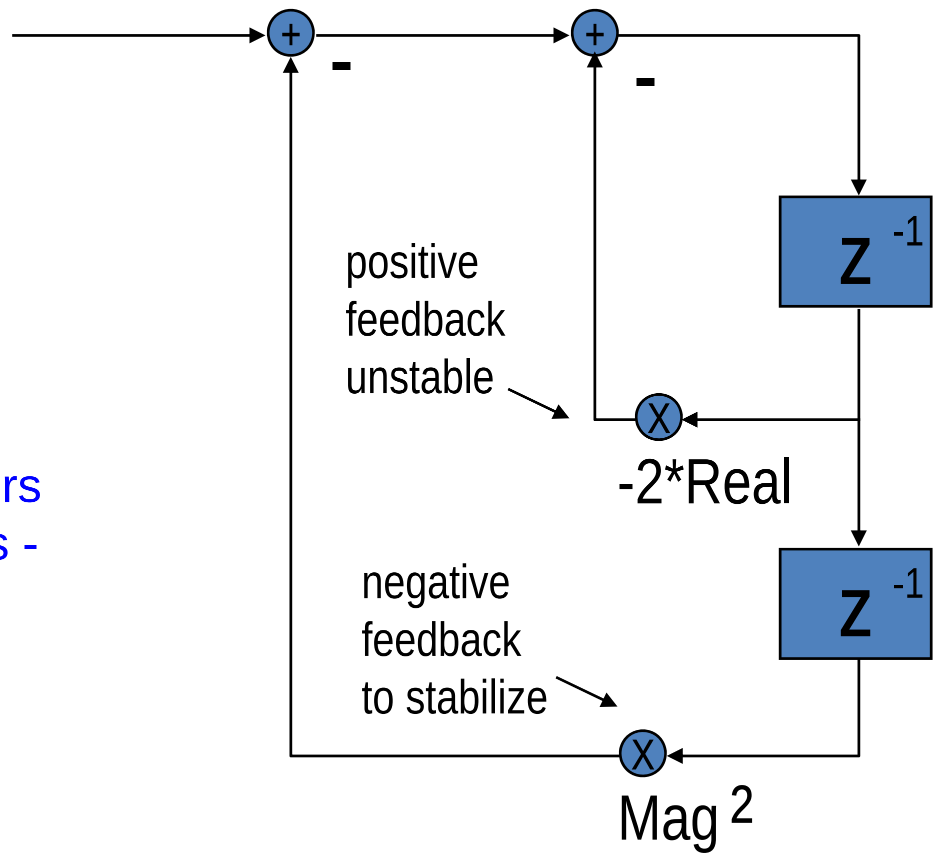
$$Y(z) = X(z)(1 + b_1 z^{-1} + b_2 z^{-2}) - Y(z)(a_1 z^{-1} + a_2 z^{-2})$$

$$y(n) = x(n) + b_1 x(n-1) + b_2 x(n-2) - a_1 y(n-1) - a_2 y(n-2)$$

For LP filter, a_1 is negative.

We therefore have positive feedback!

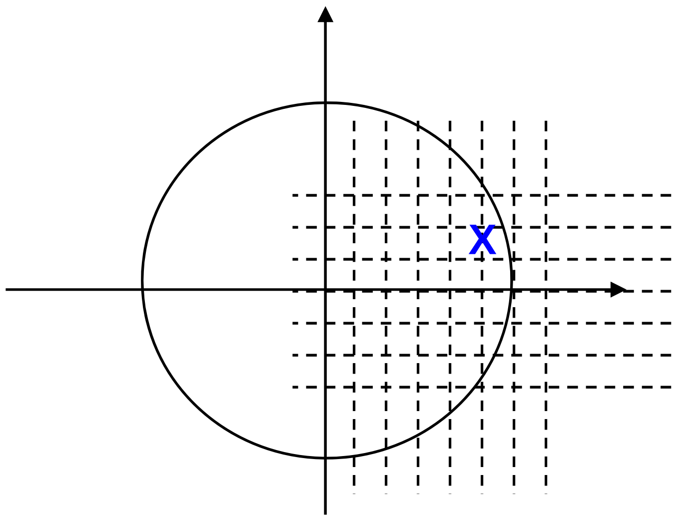
Filter Structure



2 Multipliers for 2 poles - efficient

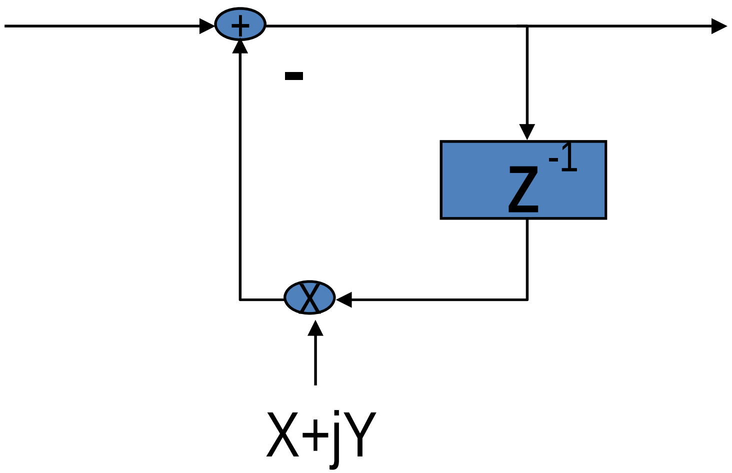
First Order Sections

- How about using first order only?
- Need complex coefficients
- Very low root sensitivity; same as for FIR
- Quantization errors are orthogonal



Orthogonal
Quantization
Errors

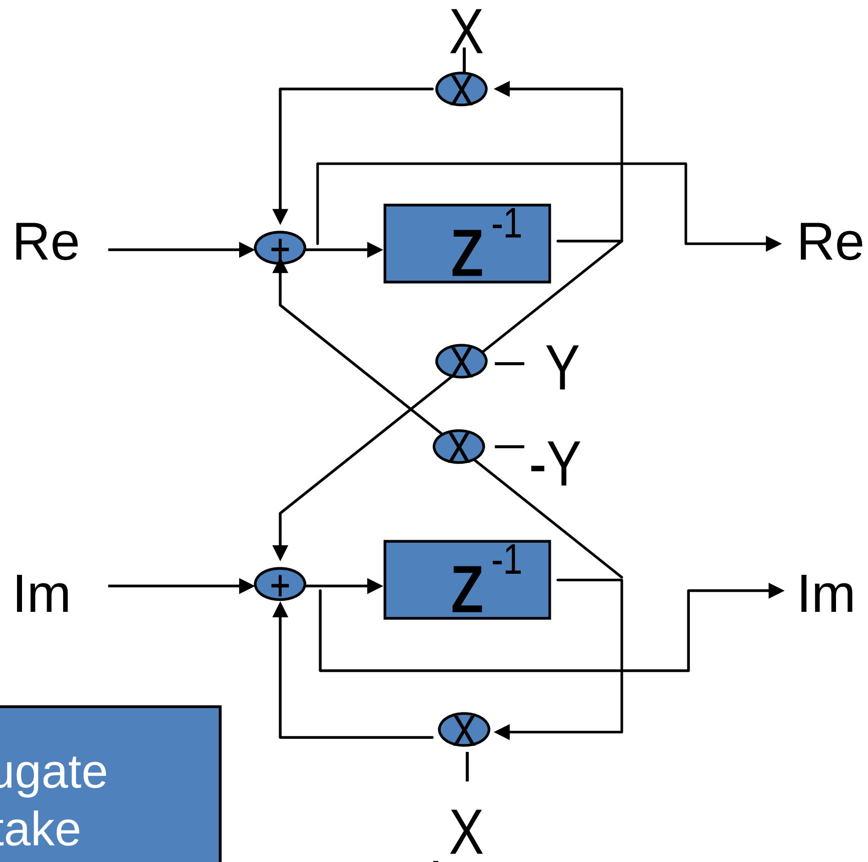
First Order Structures



4 multipliers
for two poles -
inefficient

Complex
Coefficients

To get conjugate
pair poles, take
real part of output



Real
Coefficients

Comments

- Rule of Thumb - Don’t build recursive filters.
- FIR filters probably don’t require more computation if you embed downsampling.
- Only worthwhile IIR filter is probably the leaky integrator (first order).
- Recursive filters are probably not worthwhile, apart from one notable exception - the allpass filters or microripple filters.