Estimation & Kalman Filters

(As Paul Would Say: Why the EKF is not the Eeeek!-KF)

ELEC 3004: Signals, Systems & Control

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Lecture # N [DATE]

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Hot Air…

Estimation

- An algorithm that combines *a priori* system knowledge to infer information from noisy observations

- The principle of estimation is to act as a means for reducing data to measurements

Along multiple dimensions

State Space

- We collect our set of uncertain variables into a vector …

  $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$

- The set of values that **x** might take on is termed the *state space*

- There is a *single* true value for **x**, but it is unknown

State Space Dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$H(s) = C(sI - A)^{-1}B$$

Measured versus True

- Measurement errors are inevitable

- So, add Noise to State...
  - State Dynamics becomes:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + w$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + v$$

- Can represent this as a “Normal” Distribution

$$\mathcal{N}(x;\mu,\sigma) = \frac{1}{\left(\sqrt{2\pi}\right)\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Recovering The Truth

- Numerous methods

- Termed “Estimation” because we are trying to estimate the truth from the signal

Recovering the Truth: Least Squares

- A strategy discovered by Gauss

- Least Squares in Matrix Representation

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} n & \sum_1^n t_i \\ \sum_1^n t_i & \sum_1^n t_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_1^n z_i \\ \sum_1^n t_i z_i \end{bmatrix}$$

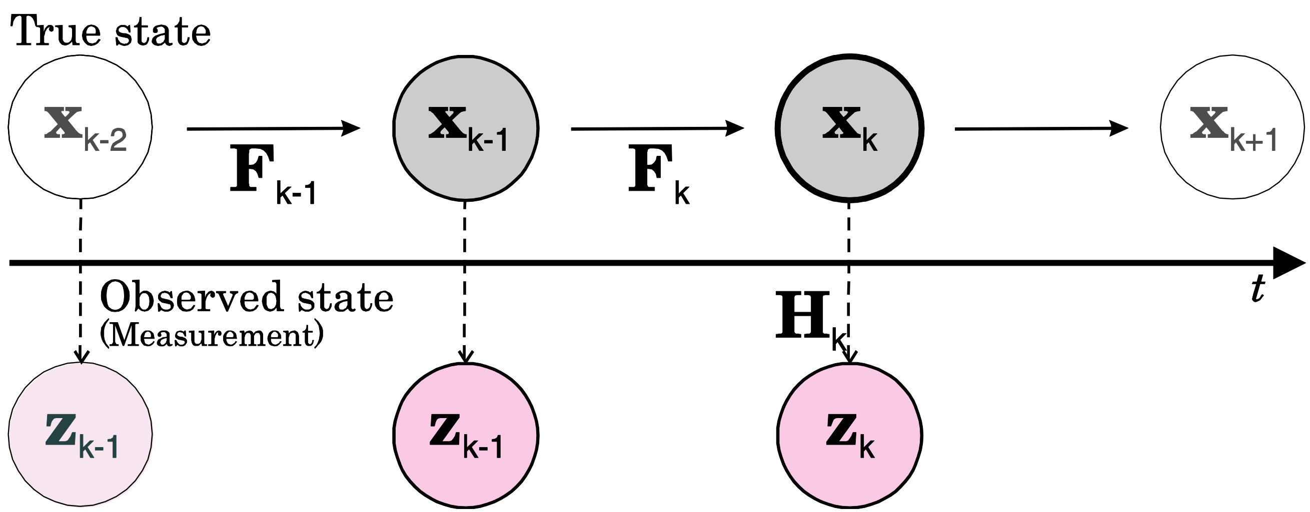
Recovering the Truth: Terminology

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} + \mathbf{w}$$

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

- $\mathbf{x}$: the state vector
- $\mathbf{x}_{A|B}$: the state of $\mathbf{x}$ at time A based on data taken up to time B
- $\hat{\mathbf{x}}$: estimate of the true state vector
- $\mathbf{F}$: system dynamics matrix in continuous time (equivalent to $\mathbf{A}$ in Eq. 1)
- $\mathbf{G}$: system control matrix relating deterministic input, $\mathbf{u}$, to the state (equivalent to $\mathbf{B}$ in Eq. 1)
- $\mathbf{H}$: measurement matrix in continuous time (equivalent to $\mathbf{C}$ in Eq. 2)
- $\mathbf{F}_i$: system model in **discrete** time at $t = t_i$
- $\mathbf{H}_i$: measurement model in **discrete** time at $t = t_i$
- $\mathbf{P}_i$: estimate covariance in **discrete** time at $t = t_i$
- $\mathbf{w}$: process uncertainty (noise) vector (of type $\mathcal{N}(0, s)$)
- $\mathbf{Q}$: process noise matrix, $\mathbf{Q} = E\left[ww^{\mathsf{T}}\right]$
- $\mathbf{Q}_i$: $\mathbf{Q}$ in discrete time at $t = t_i$
- $\mathbf{v}$: measurement noise vectors (of type $\mathcal{N}(0, \sigma)$)
- $\mathbf{R}_i$: the measurement variance matrix, $\mathbf{R} = E\left[vv^{\mathsf{T}}\right]$, in discrete time at $t = t_i$

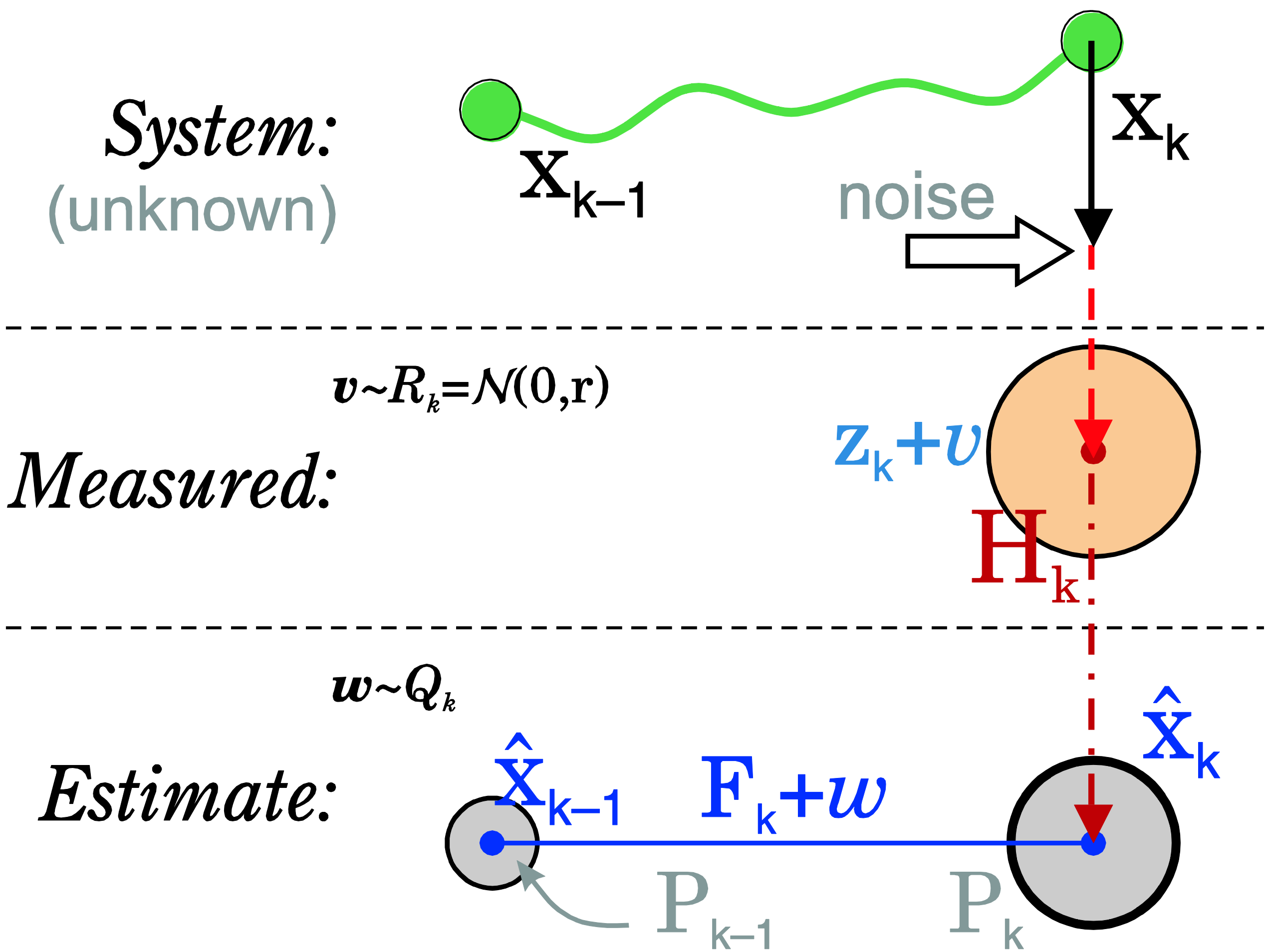
General Problem…



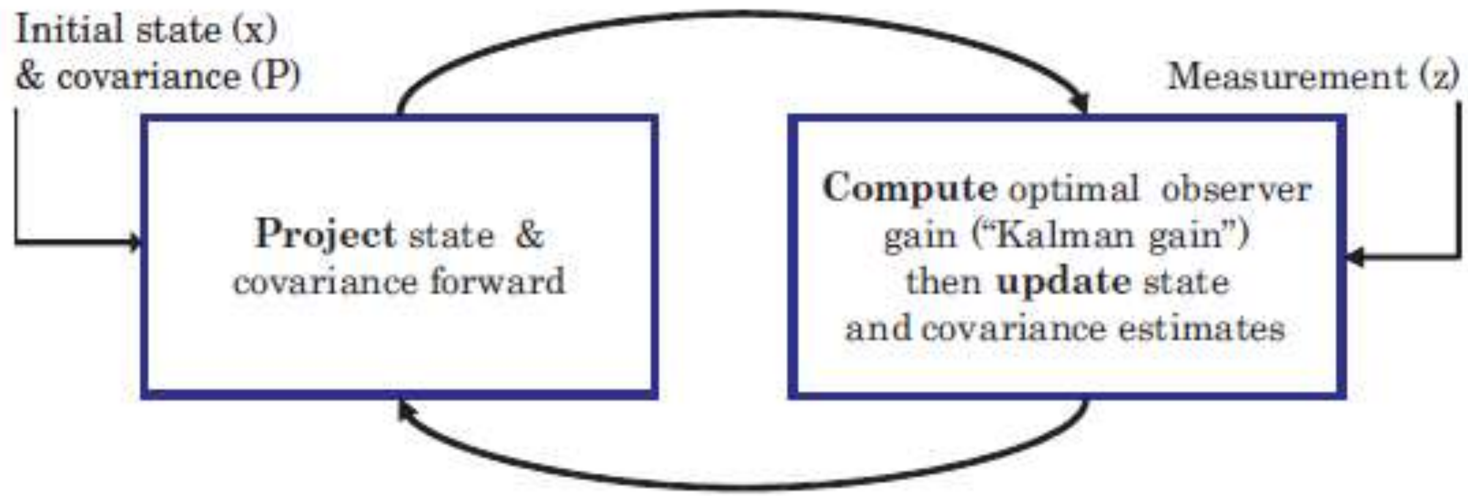
Duals and Dual Terminology

	Estimation		Control
Model:	$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$ (discrete: $\mathbf{x} = \mathbf{F}_k\mathbf{x}$)	$\leftrightarrow$	$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{A} = \mathbf{F}^{\intercal}$
Regulates:	P (covariance)	$\leftrightarrow$	M (performance matrix)
Minimized function:	Q (or $GQG^{\intercal}$)	$\leftrightarrow$	V
Optimal Gain:	K	$\leftrightarrow$	G
Completeness law:	Observability	$\leftrightarrow$	Controllability

Estimation Process in Pictures



Kalman Filter Process



KF Process in Equations

Prediction: $\hat{\mathrm{x}}_{k|k-1} = \mathbf{F}_{k-1}\hat{\mathrm{x}}_{k-1|k-1},$ (state prediction)

$\mathbf{P}_{k|k-1} = \mathbf{Q}_{k-1} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1},$ (covariance prediction)

Kalman Gain: $\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}}[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_k]^{-1},$

Update: $\mathbf{P}_{k|k} = [\mathbf{I} - \mathbf{K}_k\mathbf{H}]\mathbf{P}_{k|k-1},$ (covariance update)

$\hat{\mathrm{x}}_{k|k} = \hat{\mathrm{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\hat{\mathrm{x}}_{k|k-1})$ (state update)

KF Considerations

$$\underbrace{\hat{\mathbf{x}}_{k|k-1}}_{n\times 1} = \underbrace{\mathbf{F}_{k-1}}_{n\times n}\hat{\mathbf{x}}_{k-1|k-1} + \underbrace{\mathbf{G}_{k-1}}_{n\times j}\underbrace{\mathbf{u}_{k-1}}_{j\times 1}$$

$$\underbrace{\mathbf{P}_{k|k-1}}_{n\times n} = \underbrace{\mathbf{Q}_{k-1}}_{n\times n} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1}$$

$$\underbrace{\mathbf{K}_k}_{n\times m} = \mathbf{P}_{k|k-1}\underbrace{\mathbf{H}^{\mathsf{T}}}_{n\times m}\underbrace{\left[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_k\right]^{-1}}_{m\times m}$$

$$\mathbf{P}_{k|k} = [\mathbf{I} - \mathbf{K}_k\mathbf{H}]\mathbf{P}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\underbrace{\mathbf{z}_k}_{m\times 1} - \underbrace{\mathbf{H}}_{m\times n}\hat{\mathbf{x}}_{k|k-1} - \mathbf{H}\mathbf{G}_k\mathbf{u}_{k-1})$$

Ex: Kinematic KF: Tracking

- Consider a System with Constant Acceleration

$$\ddot{y} = -g$$
$$\dot{y} = gt + p_1$$
$$y = p_0 + p_1 t + \frac{gt^2}{2}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{F_k} = \begin{bmatrix} 0 & t_s & \frac{{t_s}^2}{2} \\ 0 & 0 & t_s \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{x}}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1})$$

In Summary

- KF:
  - The true state (*x*) is separate from the measured (*z*)
  - Lets you **combine** prior controls knowledge with measurements to filter signals and find the truth
  - It **regulates** the covariance (P)
    - As P is the scatter between *z* and *x*
    - So, if P → 0, then *z* → *x* (measurements → truth)
- EKF:
  - Takes a Taylor series approximation to get a local "F" (and "G" and "H")